Partial Orders Which Can Induce a Conjunctive Relation

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Abstract

The partial orders which can induce a conjunctive relation are characterized.

1 Introduction

Not every partial order can induce a conjunctive relation:

Theorem: Let $\prec$ be a partial order on X.

Define $f : X \to 2^X$ according to $f(x) \equiv \{y \in X; \exists z \in X\,(z \prec x \text{ and } z \prec y)\}$.

Let $\circ$ be the relation satisfying

$$x_1 \circ x_2 \Leftrightarrow \exists y \in X\,(y \prec x_1 \text{ and } y \prec x_2)\,. \tag{1.1}$$

Then $\circ$ is a conjunctive relation on X if and only if f is an injection.

Proof: ($\Rightarrow$) Suppose $\circ$ is a conjunctive relation on X. Fix $x_1, x_2 \in X$ satisfying $f(x_1) = f(x_2)$. Then

$$\{y \in X; \exists z \in X\,(z \prec x_1 \text{ and } z \prec y)\} = \{y \in X; \exists z \in X\,(z \prec x_2 \text{ and } z \prec y)\}.$$

By definition of $\circ$, this means $\{y \in X; x_1 \circ y\} = \{y \in X; x_2 \circ y\}$. This in turn means $\forall y\,(x_1 \circ y \leftrightarrow x_2 \circ y)$. Since $\circ$ is a conjunctive relation, we get $x_1 = x_2$. Thus f is an injection.

($\Leftarrow$) Suppose f is an injection. Fix $x \in X$. $x \prec x$, so there exists $y \in X$ such that $y \prec x$ and $y \prec x$, namely $y = x$, thus $x \circ x$. Next, fix $x_1, x_2 \in X$, and suppose $x_1 \circ x_2$. Then $\exists y \in X\,(y \prec x_1 \text{ and } y \prec x_2)$, so $\exists y \in X\,(y \prec x_2 \text{ and } y \prec x_1)$, thus $x_2 \circ x_1$. Finally, suppose that $\forall y \in X$, we have $x_1 \circ y \leftrightarrow x_2 \circ y$. This means $\forall y \in X$, we have $\exists z \in X\,(z \prec x_1 \text{ and } z \prec y) \leftrightarrow \exists z \in X\,(z \prec x_2 \text{ and } z \prec y)$. But this is just the condition $f(x_1) = f(x_2)$. Since f is injective, we get $x_1 = x_2$. Thus $\circ$ is a conjunctive relation.

Remark: Let $\bar{\circ}$ be the conjunctive relation induced by the partial order induced by $\circ$. Then it is NOT necessarily true that $\bar{\circ}$ and $\circ$ are identical. Indeed, take for example $X = \{1, 2, 3, 4\}$, and define $i \circ j$ according to the table below.

$\circ$	1	2	3	4
1	•	•		•
2	•	•	•	
3		•	•	•
4	•		•	•

Then the induced partial order $\prec$ is given by the table below.

$\prec$ induces the conjunctive relation $\bar{\circ}$ given by the table below,

and clearly $\circ$ and $\bar{\circ}$ are distinct.

$\prec$	1	2	3	4
1	•			
2		•		
3			•	
4				•

$\bar{\circ}$	1	2	3	4
1	•			
2		•		
3			•	
4				•

Remark: Let $\preccurlyeq$ be the partial order induced by the conjunctive relation induced by $\prec$. Then $y_1 \prec y_2$ implies $y_1 \preccurlyeq y_2$. Suppose $y_1 \prec y_2$. Fix $z \in X$ verifying $y_1 \circ z$. This means $\exists q \in X\,(q \prec y_1 \text{ and } q \prec z)$, let q be such an element. Clearly we have $q \prec y_1$, and since by assumption $y_1 \prec y_2$, the transitivity of a partial order implies $q \prec y_2$. Thus $q \prec y_2$ and $q \prec z$, whence by definition $y_2 \circ z$. Thus $\forall z \in X\,(y_1 \circ z \rightarrow y_2 \circ z)$, and therefore $y_1 \preccurlyeq y_2$, as desired.

Research Question: Is the converse true and is $\preccurlyeq$ identical to $\prec$?

2 Bibliography

Soileau, Kerry M., *Conjunctive Spaces.*