Consistency Spaces

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Abstract

We introduce the concept of a consistency space. The idea of the consistency space is motivated by the question, Given only the collection of sets of sentences which are logically consistent, is it possible to reconstruct their lattice structure?

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1 Introduction

Definition A consistency space $(\Omega, \wp)$ is comprised of a nonempty set Ω and a nonempty collection $\wp$ of subsets of Ω which satisfy the following conditions:

$$\Omega \notin \wp \tag{1.1}$$

$$\text{If } A \in \wp \text{ and } B \subseteq A, \text{ then } B \in \wp \tag{1.2}$$

Since $\wp$ is nonempty, condition 2 implies $\emptyset \in \wp$. The members of $\wp$ are called the consistent sets, the remaining subsets of Ω are called the inconsistent sets. Notice that any superset of an inconsistent set must be inconsistent.

The formulas[1] of a logic form a consistency space in a natural way. Indeed, define Ω to be the collection of formulas, and define the members of $\wp$ to be $\emptyset$ together with the subsets $S \subset \Omega$ of formulas whose logical conjunction $\bigwedge_{x \in S} x$ is not equivalent to **false**. It is easy to see that $(\Omega, \wp)$ as defined is a consistency space.

For example, consider the collection of formulas over a single Boolean variable x:

$$\Omega = \{x, \neg x\}$$

The subsets of this collection are

$$\{\emptyset, \{x\}, \{\neg x\}, \{x, \neg x\}\}$$

One possible choice for $\wp$ is $\{\emptyset, \{x\}, \{\neg x\}\}$; as defined $(\Omega, \wp)$ is a consistency space.

Next we define an equivalence relation on subsets of Ω:

Definition We say that $A \sim B$ if for every $C \subseteq \Omega$, $A \cup C \in \wp$ if and only if $B \cup C \in \wp$. In particular, if $A \sim B$, then $A \in \wp$ if and only if $B \in \wp$.

Definition We say that a consistency space $(\Omega, \wp)$ is complete if for any subset $A \subseteq \Omega$, there exists an element $x_A \in \Omega$ such that $A \sim \{x_A\}$.

[1] In the following, for technical reasons we omit **true** and **false** as formulas.

Proposition 1.1 *For any $C \subseteq \Omega$, if $A \sim B$ then $(A \cup C) \sim (B \cup C)$.*

Proof: Suppose $A \sim B$ and fix $C \subseteq \Omega$. Suppose $(A \cup C) \cup D \in \wp$. Then since $A \sim B$ and $A \cup (C \cup D) \in \wp$, it follows that $B \cup (C \cup D) \in \wp$. But this just means $(B \cup C) \cup D \in \wp$, so $(A \cup C) \cup D \in \wp$ implies $(B \cup C) \cup D \in \wp$. Similarly $(B \cup C) \cup D \in \wp$ implies $(A \cup C) \cup D \in \wp$, hence $(A \cup C) \sim (B \cup C)$.

Definition If it exists, the negation of a nonempty subset $A \in \wp$, denoted $\bar{A}$, is any member of $\wp$ satisfying the following:

$$A \cup \bar{A} \notin \wp \tag{1.3}$$

$$\text{For any } C \in \wp, \text{ if } A \cup C \notin \wp \text{ then } \bar{A} \cup C \sim C \tag{1.4}$$

$$\text{For any } C \in \wp, \text{ if } \bar{A} \cup C \notin \wp \text{ then } A \cup C \sim C \tag{1.5}$$

Proposition 1.2 *If it exists, the negation of A is unique in the sense that for any B_1 and B_2 both satisfying the criteria for $\bar{A}$, it must be true that $B_1 \sim B_2$.*

Proof: Fix nonempty $A \in \wp$. Suppose B_1 and B_2 both satisfy the criteria for $\bar{A}$. Then $B_1 \in \wp$ and $B_2 \in \wp$. By (1.3), $A \cup B_1 \notin \wp$ and $A \cup B_2 \notin \wp$. By taking $C = B_1$ in (1.4) we get that $A \cup B_1 \notin \wp$ implies $\bar{A} \cup B_1 \sim B_1$. In particular this must hold when we replace $\bar{A}$ with B_2, whence $B_2 \cup B_1 \sim B_1$.By taking $C = B_2$ in (1.4) we get that $A \cup B_2 \notin \wp$ implies $\bar{A} \cup B_2 \sim B_2$. In particular this must hold when we replace $\bar{A}$ with B_1, whence $B_1 \cup B_2 \sim B_2$. Finally, by transitivity of $\sim$ we conclude $B_1 \sim B_2$.

In the following we restrict our attention to consistency spaces $(\Omega, \wp)$ for which $\bar{A}$ exists for every $A \in \wp$.

Proposition 1.3 *For any $A \in \wp$, $\overline{(\bar{A})} \sim A$.*

<u>Proof</u>: First, by (1) we have $A \cup \bar{A} \notin \wp$ hence $\bar{A} \cup A \notin \wp$ so by (2) we have $\overline{(\bar{A})} \cup A \sim A$ and hence $A \cup \overline{(\bar{A})} \sim A$. Next, by (1) we have $\bar{A} \cup \overline{(\bar{A})} \notin \wp$ so by (3) we have $A \cup \overline{(\bar{A})} \sim \overline{(\bar{A})}$ and hence $\overline{(\bar{A})} \sim A \cup \overline{(\bar{A})}$. Finally, the transitivity of $\sim$ implies $\overline{(\bar{A})} \sim A$, as desired.

Proposition 1.4 *For any $A, B \in \wp$, $A \cup B \cup \bar{B} \notin \wp$.*

<u>Proof</u>: For a contradiction, suppose $A \cup B \cup \bar{B} \in \wp$. Since $B \cup \bar{B} \subseteq A \cup B \cup \bar{B}$, we must have $B \cup \bar{B} \in \wp$. But this is absurd, hence $A \cup B \cup \bar{B} \notin \wp$.

Proposition 1.5 *For any $A, B, C \in \wp$, if $B \cup \bar{B} \sim C$, then $A \cup \bar{C} \sim A$.*

<u>Proof</u>: We have immediately that $B \cup \bar{B} \notin \wp$. By Proposition 1.4 we have $A \cup B \cup \bar{B} \notin \wp$. Since $B \cup \bar{B} \sim C$ and $B \cup \bar{B} \cup A \notin \wp$, by Proposition 1.1 we must have $C \cup A \notin \wp$, i.e. $C \cup A \notin \wp$. By definition we have $\bar{C} \cup A \sim A$, as desired.

Proposition 1.6 *For any $A, B \in \wp$, $A \sim B$ if and only if $A \cup \bar{B} \notin \wp$ and $B \cup \bar{A} \notin \wp$.*

<u>Proof</u>: ($\Rightarrow$) Suppose $A \sim B$. Then $A \cup \bar{B} \sim B \cup \bar{B}$. Since $B \cup B \notin \wp$, it follows that $A \cup \bar{B} \notin \wp$. The claim that $B \cup \bar{A} \notin \wp$ is similarly demonstrated.

($\Leftarrow$) Suppose $A \cup \bar{B} \notin \wp$ and $B \cup \bar{A} \notin \wp$. Then $\bar{B} \cup A \notin \wp$, hence $B \cup A \sim A$, i.e. $A \cup B \sim A$. Also, $\bar{A} \cup B \notin \wp$, hence $A \cup B \sim B$. By transitivity of $\sim$, we have $A \sim B$, as desired.

Proposition 1.7 *For any $A, B \in \wp$, $A \sim B$ if and only if $\bar{A} \sim \bar{B}$.*

Proof: ($\Rightarrow$) Suppose $A \sim B$. Then by Proposition 1.6, we have $A \cup \bar{B} \notin \wp$ and $B \cup \bar{A} \notin \wp$. By definition, $A \cup \bar{B} \notin \wp$ implies $\bar{A} \cup \bar{B} \sim \bar{B}$. By the same definition, $B \cup \bar{A} \notin \wp$ implies $\bar{B} \cup \bar{A} \sim \bar{A}$. Since $\bar{A} \cup \bar{B} \sim \bar{B} \cup \bar{A}$, it follows by transitivity that $\bar{A} \sim \bar{B}$.

($\Leftarrow$) Suppose $\bar{A} \sim \bar{B}$. Then the immediately previous argument implies $\overline{(\bar{A})} \sim \overline{(\bar{B})}$. Since $\overline{(\bar{A})} \sim A$ and $\overline{(\bar{B})} \sim B$ by Proposition 1.3, transitivity of $\sim$ implies $A \sim B$.

Definition If $\bar{B}$ exists, we say that $A \to B$ if and only if $A \cup \bar{B} \notin \wp$.

Proposition 1.8 *For any $A \in \wp$, $A \to A$.*

Proof: Since by definition we have that $A \cup \bar{A} \notin \wp$, by definition it follows that $A \to A$.

Proposition 1.9 *For any $A, B \in \wp$, $A \sim B$ if and only if $A \to B$ and $B \to A$.*

Proof: ($\Rightarrow$) Suppose $A \sim B$. By the definition of $\sim$ it follows that $A \cup \bar{B} \sim B \cup \bar{B}$. By definition we have $B \cup \bar{B} \notin \wp$, thus by the definition of $\sim$ it follows that $A \cup \bar{B} \notin \wp$ and thus $A \to B$. Since $A \sim B$, it follows by the definition of $\sim$ that $A \cup \bar{A} \sim B \cup \bar{A}$. By definition we have $A \cup \bar{A} \notin \wp$, thus by the definition of $\sim$ it follows that $B \cup \bar{A} \notin \wp$ and thus $B \to A$.

($\Leftarrow$) Suppose $A \to B$ and $B \to A$. Then by definition, $A \cup \bar{B} \notin \wp$ and $B \cup \bar{A} \notin \wp$, hence by Proposition 1.6, we have $A \sim B$.

Proposition 1.10 *For any $A, B, C \in \wp$, if $A \to B$ and $B \to C$, then $A \to C$.*

Proof: Suppose $A \to B$ and $B \to C$. Since $A \to B$, we have by definition that $A \cup \bar{B} \notin \wp$, i.e. $\bar{B} \cup A \notin \wp$, which by definition implies $B \cup A \sim A$. The definition of $\sim$ implies $B \cup A \cup \bar{C} \sim A \cup \bar{C}$. But $B \to C$, hence $B \cup \bar{C} \notin \wp$ by definition, and then immediately $B \cup A \cup \bar{C} \notin \wp$ since $B \cup \bar{C} \subseteq B \cup A \cup \bar{C}$. Since $B \cup A \cup \bar{C} \sim A \cup \bar{C}$, this implies $A \cup \bar{C} \notin \wp$. But this means $A \to C$.

Proposition 1.11 *For any $A, B, C, Q \in \wp$, if $Q \to A$, $Q \to B$, and $A \cup B \sim C$, then $Q \to C$.*

Proof: Suppose $Q \to A$ and $Q \to B$, and $A \cup B \sim C$. Then $Q \cup \bar{A} \notin \wp$ and $Q \cup \bar{B} \notin \wp$ by definition. By definition, $Q \cup \bar{A} \notin \wp$ implies $Q \cup A \sim Q$, and $Q \cup \bar{B} \notin \wp$ implies $Q \cup B \sim Q$, so $A \cup B \cup Q \sim A \cup Q \sim Q$, and thus by transitivity we have $A \cup B \cup Q \sim Q$. By definition of equivalence, this means $Q \cup \bar{C} \sim A \cup B \cup Q \cup \bar{C}$. Since $A \cup B \sim C$, we have $A \cup B \cup Q \cup \bar{C} \sim C \cup Q \cup \bar{C} \supseteq \bar{C} \cup C \notin \wp$, hence $A \cup B \cup Q \cup \bar{C} \notin \wp$, thus since $Q \cup \bar{C} \sim A \cup B \cup Q \cup \bar{C}$, we have, again by definition of equivalence, that $Q \cup \bar{C} \notin \wp$ and thus $Q \to C$, as desired.

Proposition 1.12 *For any $A, B, C, Q \in \wp$, if $A \to Q$, $B \to Q$, and $\bar{A} \cup \bar{B} \sim \bar{C}$ then $C \to Q$.*

Proof: Suppose $A \to Q$ and $B \to Q$, and $\bar{A} \cup \bar{B} \sim \bar{C}$. Then by definition we have $A \cup \bar{Q} \notin \wp$ and $B \cup \bar{Q} \notin \wp$. By definition, $A \cup \bar{Q} \notin \wp$ implies $\bar{A} \cup \bar{Q} \sim \bar{Q}$, and $B \cup \bar{Q} \notin \wp$ implies $\bar{B} \cup \bar{Q} \sim \bar{Q}$. Since $\bar{A} \cup \bar{Q} \sim \bar{Q}$, by definition of equivalence we have $\bar{A} \cup \bar{B} \cup \bar{Q} \sim \bar{Q} \cup \bar{B}$. Since $\bar{Q} \cup \bar{B} \sim \bar{Q}$, this implies $\bar{A} \cup \bar{B} \cup \bar{Q} \sim \bar{Q}$. By definition of equivalence, this gives $C \cup \bar{Q} \sim C \cup \bar{A} \cup \bar{B} \cup \bar{Q}$. Since $\bar{A} \cup \bar{B} \sim \bar{C}$, again by definition of equivalence we get $C \cup \bar{A} \cup \bar{B} \cup \bar{Q} \sim C \cup \bar{C} \cup \bar{Q}$. Combining these results yields $C \cup \bar{Q} \sim C \cup \bar{C} \cup \bar{Q}$. Since $C \cup \bar{C} \cup \bar{Q} \supseteq C \cup \bar{C} \notin \wp$, it follows that $C \cup \bar{C} \cup \bar{Q} \notin \wp$. Since $C \cup \bar{Q} \sim C \cup \bar{C} \cup \bar{Q}$, we get that $C \cup \bar{Q} \notin \wp$ and thus $C \to Q$. Thus in a sense C is a least upper bound of A and B.

Definition If there exists some C such that $\bar{A} \cup \bar{B} \sim \bar{C}$, we define the join $A \vee B = C$.

Definition If there exists some C such that $A \cup B \sim C$, we define the meet $A \wedge B = C$.

Proposition 1.13 *For any $A, B \in \wp$, $A \to B$ if and only if $\bar{B} \to \bar{A}$.*

Proof: ($\Rightarrow$) Suppose $A \to B$. Then by definition, $A \cup \bar{B} \notin \wp$, hence $\bar{B} \cup A \notin \wp$, and $\bar{B} \cup \overline{(\bar{A})} \notin \wp$ by Proposition 1.7 and the definition of equivalence. But this means $\bar{B} \to \bar{A}$.

($\Leftarrow$) Suppose $\bar{B} \to \bar{A}$. Then by definition, $\bar{B} \cup \overline{(\bar{A})} \notin \wp$, hence $\overline{(\bar{A})} \cup \bar{B} \notin \wp$ and $A \cup \bar{B} \notin \wp$ by Proposition 1.7 and the definition of equivalence. But this means $A \to B$.

Proposition 1.14 *For any $A, B, C, D, Q \in \wp$, if $A \to B$, $Q \cup A \sim C$ and $Q \cup B \sim D$, then $C \to D$.*

Proof: Since $A \to B$, by definition we have $A \cup \bar{B} \notin \wp$, and thus $B \cup A \sim A$ by definition. The definition of equivalence implies $Q \cup B \cup A \sim Q \cup A$. Since $B \cup A \sim A$, equivalence implies $Q \cup A \cup \bar{D} \sim Q \cup B \cup A \cup \bar{D}$. Since $Q \cup B \sim D$ and thus equivalence implies $Q \cup B \cup \bar{D} \sim D \cup \bar{D}$; by definition, $D \cup \bar{D} \notin \wp$ so we get $Q \cup B \cup \bar{D} \notin \wp$. Since $Q \cup B \cup A \cup \bar{D} \supseteq Q \cup B \cup \bar{D}$, we get $Q \cup B \cup A \cup \bar{D} \notin \wp$. Since $Q \cup A \cup \bar{D} \sim Q \cup B \cup A \cup \bar{D}$, this implies $Q \cup A \cup \bar{D} \notin \wp$. Since $Q \cup A \sim C$, and thus by equivalence $Q \cup A \cup \bar{D} \sim C \cup \bar{D}$, hence $Q \cup A \cup \bar{D} \notin \wp$ implies $C \cup \bar{D} \notin \wp$. But this just means that $C \to D$, as desired.

2 Illustrative Counterexamples

Counterexample: Not every member of $\wp$ gas a negation. Indeed, take $\Omega = \{a, b\}$ and $\wp$

Counterexample: If $\forall i \in I, A_i \sim B_i$, $\bigcup_{i \in I} A_i \in \wp$ and $\bigcup_{i \in I} B_i \in \wp$, then it does not necessarily follow that $\bigcup_{i \in I} A_i \sim \bigcup_{i \in I} B_i$. Indeed, take Ω to be the integers and take

$$\wp = \{\emptyset\} \cup \left\{ N \subseteq \Omega; N \neq \emptyset, \sum_{n \in N} z^n \text{ converges at at least one point in } \mathbb{C} \right\}.$$

Define $I = \{1, 2, 3, \cdots\}$, $A_i = \{1, 2, 3, \cdots, i\}$ and

$B_i = \{-1, -2, -3, \cdots, -i\}$. Next, $\forall i \in I, A_i \sim B_i$. $\bigcup_{i \in I} A_i \in \wp$ because $\sum_{n=1}^{\infty} z^n$ converges at $z = 0$. $\bigcup_{i \in I} B_i \in \wp$ because $\sum_{n=1}^{\infty} z^{-n}$ converges at $z = 2$. However, if we take $C = \{1, 2, 3, \cdots\}$, then $\bigcup_{i \in I} A_i \cup C = \bigcup_{i \in I} A_i \in \wp$, yet $\bigcup_{i \in I} B_i \cup C = \{\cdots, -3, -2, -1, 1, 2, 3, \cdots\} \notin \wp$ since $\cdots + z^{-3} + z^{-2} + z^{-1} + z + z^2 + z^3 + \cdots$ does not converge for any $z \in C$.

Thus it is not true that $\bigcup_{i \in I} A_i \sim \bigcup_{i \in I} B_i$.

Counterexample: One might ask, is the union of a nested collection of consistent sets always consistent? The answer is no. Consider the sets of statements $A_n = \{$"A is an integer $\geqslant$ i"$; 1 \leqslant i \leqslant n\}$ for $n = 1, 2, 3, \cdots$ Clearly each $A_n \in \wp$ and $A_n \subset A_{n+1}$. However, it is clear that $\bigcup_{n=1}^{\infty} A_n$ is always inconsistent, since no integer is greater than every other. Hence $\bigcup_{n=1}^{\infty} A_n \notin \wp$.

3 Bibliography

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