A Construction of the Null Set

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Abstract

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1 Introduction

The imaginary number i satisfies the relation $i^2 = -1$, but this information falls short of a construction. Accordingly, one constructs i by identifying it with the quotient of the ideal (X^2+1) in $\mathbb{R}[X]$, the ring of real polynomials in X. This produces a natural injection of the reals, extending them to include i; this injection maps each real r to the element $r + (X^2+1)$, the coset of r in $\mathbb{R}[X]$. The purely imaginary numbers of the form ci correspond to the coset $cX + (X^2+1)$, and thus each complex number $r+ci$ corresponds to the coset $r + cX + (X^2+1)$. This fully constructs the complex numbers as an extension of the reals.

It seems that defining the null set, say, as the unique set that is a subset of every set, or the set that has no members, also falls short of a construction. We propose the following:

Let U be any set with at least one member. We begin with the Cartesian product $2^U \times 2^U$ of subsets of U, and define an equivalence relation on these pairs of subsets. Indeed, we say that $(A, B) \equiv (C, D)$ if and only if $x \in A$ and $x \in B \Leftrightarrow x \in C$ and $x \in D$. To put it another way, either A and B have no members in common and neither do C and D, or A and B have members in common, C and D have members in common, and $A \cap B = C \cap D$.

We now observe that we can embed every subset S containing at least one member into the collection of equivalence classes as follows: Let $\overline{(E, F)}$ denote the equivalence class containing the pair (E, F). For pairs (E, F) such that E and F have members in common, we define the mapping f according to $f : S \mapsto \overline{(S, S)}$. This mapping is injective on the collection of sets with at least one member; indeed, suppose $f(S_1) = f(S_2)$, then $\overline{(S_1, S_1)} = \overline{(S_2, S_2)}$, thus $(S_1, S_1) \equiv (S_2, S_2)$, which gives $S_1 \cap S_1 = S_2 \cap S_2$, i.e. $S_1 = S_2$. The remaining equivalence class, the one corresponding to pairs of sets with no elements in common is then defined to represent the empty set; we denote this equivalence class by $\emptyset$.

The final step is to establish a homomorphism that respects basic set operations. We define

1. $\overline{(S_1, S_1)} \cap \overline{(S_2, S_2)} \equiv \overline{(S_1, S_2)}$ for any S_1, S_2 with elements in common, and $\overline{(S_1, S_1)} \cap \overline{(S_2, S_2)} \equiv \emptyset$ otherwise.

2. $\overline{(S, S)} \cap \emptyset \equiv \emptyset$ and $\overline{(S, S)} \cup \emptyset \equiv \overline{(S, S)}$ for any nonempty S.

3. $\overline{(S_1, S_1)} \cup \overline{(S_2, S_2)} \equiv \overline{(S_1 \cup S_2, S_1 \cup S_2)}$ for any nonempty S_1, S_2; in par-

ticular, $\overline{(S_1, S_1)} \cup \overline{(U, U)} \equiv \overline{(U, U)}$

4. $\overline{(S, S)}^c \equiv \overline{(S^c, S^c)}$ for any nonempty $S \neq U$, and $\overline{(U, U)}^c \equiv \emptyset$

5. $\overline{(S_1, S_1)} \subseteq \overline{(S_2, S_2)} \Leftrightarrow S_1 \subseteq S_2$ for any nonempty S_1, S_2; we stipulate $\emptyset \subseteq \overline{(S, S)}$ for any nonempty S, and $\emptyset \subseteq \emptyset$.

2 Bibliography

[1] Boole, George (1848), The Calculus of Logic, *The Cambridge and Dublin Mathematical Journal*, vol. 3.

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