The Prom Date

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Abstract

We consider the problem of being presented with up to n options in sequence, and determining when to stop and accept the latest option. Once rejected, an option will not again be offered. Each option has a known probability of being offered, but nothing is known about the sequence in which options will actually be offered. Each option has a known, finite positive value. We present an algorithm which maximizes the expected value of the option eventually accepted.

Keywords. optimal, stopping, probability, algorithm, expected value.

1 Introduction

Suppose a young lady is eagerly anticipating the upcoming prom. Of course, she will need a date. For each of the n young men whom she thinks might ask her, she has estimated the probability that he will ask, together with his desirability, with the most desirable boy having desirability equal to 1, and every boy having a desirability greater than 0. These data are represented by the variables $p_1, p_2, p_3, \ldots, p_{n-1}, p_n$ and $x_1, x_2, x_3, \ldots, x_{n-1}, x_n$, respectively ($x_n = 1$ by assumption), with the indexing indicating desirability ordering, i.e. $i < j$ implies $x_i \leqslant x_j$. Since she has included only boys whom she thinks might ask her, all of the p_i are positive. She doesn't know for certain which boys are willing to ask her. By "the boys who are willing to ask her," we mean the set of boys who would ask her if she rejected every boy who asked. She assumes that the boys who are willing to ask will do so in no particular order and that each possible ordering of these boys' invitations is equally likely. Once she refuses a boy, he will not ask again. If she happens to refuse every boy that asks, she will end up with no date for the prom, a payoff of 0. If $p_i < 1$ for every i, it's possible that no boy will ask, yielding her a payoff of 0. In this paper we assume that her objective is to maximize the expected desirability of her eventual date.

2 The First-ask-probability

In the following we will make use of a formula giving the probability that among the boys who have not asked, a particular boy will be the first among them to ask.

For probabilities and desirabilities $\{p_i\}_{i=1}^n$ and $\{x_i\}_{i=1}^n$,

$$P(k \text{ is the first boy to ask})$$

$$= p_k \left(1 - \frac{1}{2} \sum_{(j_1) \in M_{n,k,1}} p_{j_1} + \cdots + \frac{(-1)^{n-1}}{n} \sum_{(j_1, j_2, \ldots, j_{n-1}) \in M_{n,k,n-1}} p_{j_1} p_{j_2} \ldots p_{j_{n-1}} \right),$$

where

$$M_{n,k,r} = \{(j_1, j_2, \ldots, j_r); \{j_1, j_2, \ldots, j_r\} \subseteq \{1, 2, \ldots n\} \backslash \{k\}; j_1 < j_2 < \cdots < j_r\}.$$

For example, for $n = 2$,

$$P(1 \text{ is the first boy to ask}) = p_1 \left(1 - \frac{1}{2} \sum_{(j_1) \in \{(2)\}} p_{j_1} \right) = p_1 \left(1 - \frac{p_2}{2} \right)$$

and

$$P(2 \text{ is the first boy to ask}) = p_2 \left(1 - \frac{1}{2} p_1 \right) = p_2 \left(1 - \frac{p_1}{2} \right).$$

For $n = 3$,

$$P(1 \text{ is the first boy to ask})$$

$$= p_1 \left(1 - \frac{1}{2} \sum_{(j_1) \in \{(j_1); \{j_1\} \subseteq \{2,3\}\}} p_{j_1} + \frac{1}{3} \sum_{(j_1, j_2) \in \{(2,3)\}} p_{j_1} p_{j_2} \right),$$

$$= p_1 \left(1 - \frac{1}{2} (p_2 + p_3) + \frac{1}{3} p_2 p_3 \right),$$

For convenience, in the following, we write

$$f_{n,k} (p_1, p_2, \ldots p_n) \equiv P(k \text{ is the first boy to ask})$$

$$= p_k \left(1 - \frac{1}{2} \sum_{(j_1) \in M_{n,k,1}} p_{j_1} + \cdots + \frac{(-1)^{n-1}}{n} \sum_{(j_1, j_2, \ldots, j_{n-1}) \in M_{n,k,n-1}} p_{j_1} p_{j_2} \cdots p_{j_{n-1}} \right),$$

3 An Algorithm

We propose an algorithm for deciding whether to accept or reject the invitation of a given boy. We regard the girl's problem as playing a sequence of games. The first game presents a boy who has asked her, and $n-1$ who have not. She has two available plays in this game: accept or reject his invitation. If she accepts, the game is over and her payoff is equal to the desirability of the accepted boy. If she rejects him, she is presented with a new game. This game now consists of one of the remaining boys as the asker, and the $n-2$ boys who haven't asked. We assume that the desirability values for each of the $n-1$ boys are unchanged. However, as we will see below, this is generally not the case for the ask-probability values. She continues playing this sequence of games, each with fewer boys than the previous one, doing the appropriate Bayesian update on the probabilities when passing from one game to the next. This continues until either she accepts some boy or no boy remains who is willing to ask her.

To see why each boy's ask-probability will, in general, vary from game to game in the sequence, we give a simple example. Assuming the ask-probability of each of the boys is $\frac{1}{2}$, the relative probabilities of any possible series of invitations are represented by the entries in the table below. That is, the probability of a series of invitations is proportional to the number of its appearances in the table. ∅ is the case in which no boys ask, 12 means boy 1 asks first, then boy 2, but boy 3 doesn't ask, and so on.

∅	1	2	3	12	21	31	123
∅	1	2	3	12	21	31	132
∅	1	2	3	12	21	31	213
∅	1	2	3	13	23	32	231
∅	1	2	3	13	23	32	312
∅	1	2	3	13	23	32	321

Now suppose, for instance, that boy 2 asks first. The table

		2			21		
		2			21		
		2			21		213
		2			23		231
		2			23		
		2			23		

represents the possibilities, given that boy 2 asked first, for the set of boys who are willing to ask and in what order they would ask. The three entries of 21 and the entries 213 and 231 represent the outcome that boy 1 would eventually ask, for a total of 5 entries. Dividing that by the total of 14 equally likely outcomes yields a conditional probability of $\frac{5}{14}$. Thus boy 2 having asked first has slightly altered the probability that boy 1 would eventually ask.

3.1 Specifics

Each time a boy, say boy k, asks, she determines the largest expected desirability of the boys who haven't asked and compares this number with x_k. Explicitly, she accepts boy k if and only if

$$x_k \geqslant \max\{p_{j|k}x_j; \text{boy } j \text{ hasn't asked}\}.$$

Here $p_{j|k}$ denotes the probability that boy j is willing to ask, given that boy k is the first boy to ask. Note that in general, $p_{j|k} \neq p_j$, since the observation of boy k as the first boy to ask alters the effective asking probabilities of the remaining boys.

With this in mind, we see that if she rejects boy k, the remaining $n-1$ boys form, after the appropriate reindexing of the desirabilities and Bayesian update of the probabilities, a new problem of the same type, and the first of these boys to ask can be accepted or refused according to a reapplication of the above criterion. This iteration continues until some boy is accepted via the criterion, or all boys who are willing to ask are refused.

This can be illustrated by the following simple example. Let $n = 2$, with data $\{p_1, p_2\}$ and $\{x_1, 1\}$, with $x_1 < \frac{p_2}{2-p_2}$. Suppose boy 1 has asked.

$$P(\text{boy 2 will ask}|\text{boy 1 asked first})$$
$$= \frac{P(\text{boy 2 will ask and boy 1 asked first})}{P(\text{boy 1 asked first})} = \frac{\frac{1}{2}p_1p_2}{p_1 - \frac{1}{2}p_1p_2} = \frac{p_2}{2-p_2}$$

Because the probability that boy 2 will subsequently ask is $\frac{p_2}{2-p_2}$, in this example she gets a payoff of 1 if boy 2 does ask, and 0 if not, so the expected value of boy 2's invitation is $\frac{p_2}{2-p_2} \cdot 1 + \left(1 - \frac{p_2}{2-p_2}\right) \cdot 0 = \frac{p_2}{2-p_2}$. Since $x_1 < \frac{p_2}{2-p_2}$, boy 1 is rejected.

3.2 Bayesian Update

As mentioned before, the first boy that asks her will alter the probabilities that the other boys will ask her eventually.

Proposition 3.1 *The updated probability that boy j would eventually ask, given that boy j has not yet asked and boy k asked first, is given by*

$$p_{j|k} = \frac{f_{n,k}\left(p_1, p_2, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n\right)}{f_{n,k}\left(p_1, p_2, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_n\right)} p_j.$$

Proof

$$P(\text{boy } j \text{ would eventually ask}|\text{boy } k \text{ asked first})$$
$$= \frac{P(\text{boy } j \text{ would eventually ask and boy } k \text{ asked first})}{P(\text{boy } k \text{ asked first})}$$
$$= \frac{P(\text{boy } k \text{ asked first and boy } j \text{ would eventually ask})}{P(\text{boy } k \text{ asked first})}$$

$$= \frac{P(\text{boy } k \text{ asked first}|\text{boy } j \text{ would eventually ask})P(\text{boy } j \text{ would eventually ask})}{P(\text{boy } k \text{ asked first})}.$$

Since $P(\text{boy } k \text{ asked first}|\text{boy } j \text{ would eventually ask})$

$$= f_{n,k}\left(p_1, p_2, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n\right),$$

$$P(\text{boy } j \text{ would eventually ask}) = p_j,$$

and $P(\text{boy } k \text{ asked first}) = f_{n,k}\left(p_1, p_2, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_n\right)$, we get that

$$P(\text{boy } j \text{ would eventually ask}|\text{boy } k \text{ asked first})$$

$$= \frac{f_{n,k}\left(p_1, p_2, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n\right)}{f_{n,k}\left(p_1, p_2, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_n\right)} p_j.$$

3.3 A Detailed Example

We conclude with an example. Let the boys' data be as in the following table:

Boy	x_i	p_i
Al	.1	.9000
Ben	.2	.7000
Carl	.7	.4000
Don	1	.1000

Suppose the first boy to ask her (in this fictitious scenario) is Ben. Having observed Ben asking first, she does a Bayesian update of the probabilities for Al, Carl and Don later asking and obtains

Boy	x_i	p'_i	Expected Value
Al	.1	.8320	.0832
Carl	.7	.3001	.2100
Don	1	.0690	.0690

Note that Ben's desirability is .2, which is less than Carl's expected value, so she rejects Ben's invitation.

We now begin anew, with the boys, desirabilities and probabilities given in the previous table. The next boy to ask her is Al. Having observed Al asking first in this new game, she does a Bayesian update of the probabilities for Carl and Don later asking and obtains

Boy	x_i	p''_i	Expected Value
Carl	.7	.1782	.1247
Don	1	.0377	.0377

Note that Al's desirability is .1, which is less than Carl's expected value, so she rejects Al's invitation.

Again we begin anew, with the boys, desirabilities and probabilities given in the previous table. The next boy to ask her is Carl. Having observed Carl asking first, she does a Bayesian update of the probability for Don later asking and obtains

Boy	x_i	p'''_i	Expected Value
Don	1	.0192	.0192

Note that Carl's desirability is .7, which is greater than the expected values of all the boys who haven't asked her (in this case, just Don), so she accepts Carl's invitation.

4 Bibliography

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