Scoring Metrics on Separable Metric Spaces

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Abstract

We define scoring metrics on separable metric spaces and show that they are always no coarser than the metrics from which they spring.

Keywords. coarser, dense, metric, scoring, separable, sequence, topology.

1 Introduction

If one tries to imagine the "simplest" possible metric on a given set X, an argument could be made for the trivial metric $\tau_c(x,y) = \begin{cases} c & x \neq y \\ 0 & x = y \end{cases}$, with $c > 0$. What is next simplest? We propose the scoring metric. We first envision "delegate functions" $f_n : X \to X$ which associate to each point $x \in X$ a point $f_n(x) \in X$. For any two points $x, y \in X$, for each $n = 1, 2, 3, \cdots$ one may compute the score $\tau_{a_n}(f_n(x), f_n(y)) = \begin{cases} a_n & f_n(x) \neq f_n(y) \\ 0 & f_n(x) = f_n(y) \end{cases}$, where $\{a_n\}_{n=1}^{\infty}$ is a non-increasing sequence of positive real numbers such that $\sum_{i=1}^{\infty} a_i$ converges. These scores are summed to produce the scoring function

$\rho(x,y) = \sum_{i=1}^{\infty} \tau_{a_i}\left(f_i(x), f_i(y)\right)$. In the following we show that if X is a separable metric space, it is straightforward to find functions $f_n : X \to X$ and sequences $\{a_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} \tau_{a_i}\left(f_i(x), f_i(y)\right)$ is a metric on X, and that the induced topology is no coarser than the original topology.

2 Propositions

Let X be a separable metric space with metric $\sigma(\cdot,\cdot)$. Let $\{r_i\}_{i=1}^{\infty}$ be a countable dense subset of X with $r_i = r_j$ only if $i = j$. Define $f_n : X \to X$ as follows:

$$f_1(x) = r_1, \tag{2.1}$$

and

$$f_n(x) = \begin{cases} f_{n-1}(x) & \text{if } \sigma\left(x, f_{n-1}(x)\right) \leqslant \sigma(x, r_n) \\ r_n & \text{if } \sigma(x, r_n) < \sigma\left(x, f_{n-1}(x)\right) \end{cases} \tag{2.2}$$

for $n > 1$.

Note that

$$\sigma(x, f_n(x)) = \begin{cases} \sigma(x, f_{n-1}(x)) & \text{if } \sigma\left(x, f_{n-1}(x)\right) \leqslant \sigma(x, r_n) \\ \sigma(x, r_n) & \text{if } \sigma(x, r_n) < \sigma\left(x, f_{n-1}(x)\right) \end{cases} \tag{2.3}$$
$$= \min(\sigma(x, f_{n-1}(x)), \sigma(x, r_n)),$$

hence

$$\sigma(x, f_n(x)) \leqslant \sigma(x, r_n). \tag{2.4}$$

Fix $\epsilon > 0$. Since $\{r_i\}_{i=1}^{\infty}$ is dense, we can find $N \geqslant 1$ such that $\sigma(x, r_N) < \epsilon$. Since $\sigma(x, f_N(x)) \leqslant \sigma(x, r_N)$, it follows that $\sigma(x, f_N(x)) < \epsilon$. By induction on $\sigma(x, f_n(x)) \leqslant \sigma(x, f_{n-1}(x))$ we get that $\sigma(x, f_n(x)) < \epsilon$ for all $n \geqslant N$. Thus $\lim\limits_{n\to\infty} f_n(x) = x$.

Let $\{a_n\}_{n=1}^{\infty}$ be a non-increasing sequence of positive real numbers such that $\sum\limits_{i=1}^{\infty} a_i$ converges. Then define $\rho : X \times X \to \mathbb{R}$ as

$$\rho(x, y) \equiv \sum_{i=1}^{\infty} \tau_{a_i} \left(f_i(x), f_i(y) \right). \tag{2.5}$$

$\rho(\cdot, \cdot)$ is well-defined and finite because it is dominated by $\sum\limits_{i=1}^{\infty} a_i < \infty$.

We claim that $\rho(\cdot, \cdot)$ is a metric over X, because of the following three observations:

1. $$\rho(x, x) = \sum_{i=1}^{\infty} \tau_{a_i} \left(f_i(x), f_i(x) \right) = \sum_{i=1}^{\infty} 0 = 0; \tag{2.6}$$

2. If $x, y \in X$ and $x \neq y$, then given any integer N, there exists $n > N$ such that $f_n(x) \neq f_n(y)$, since $\lim\limits_{n\to\infty} f_n(x) = x$ and $\lim\limits_{n\to\infty} f_n(y) = y$.. Hence $\rho(x, y) > 0$ whenever $x \neq y$;

3. Fix any three points $x, y, z \in X$ and any positive integer i. If $f_i(x) = f_i(z)$ it follows that $\tau_{a_i} \left(f_i(x), f_i(z) \right) = 0 \leqslant \tau_{a_i} \left(f_i(x), f_i(y) \right) + \tau_{a_i} \left(f_i(y), f_i(z) \right)$. If $f_i(x) \neq f_i(z)$, we may infer that either $f_i(x) \neq f_i(y)$ or $f_i(y) \neq f_i(z)$, hence $\tau_{a_i} \left(f_i(x), f_i(y) \right) = a_i$ or $\tau_{a_i} \left(f_i(y), f_i(z) \right) = a_i$, which in turn implies $\tau_{a_i} \left(f_i(x), f_i(y) \right) + \tau_{a_i} \left(f_i(y), f_i(z) \right) \geqslant a_i \geqslant \tau_{a_i} \left(f_i(x), f_i(z) \right)$. Hence

in either case $\tau_{a_i}(f_i(x), f_i(y)) + \tau_{a_i}(f_i(y), f_i(z)) \geqslant \tau_{a_i}(f_i(x), f_i(z))$. Next recall that $\rho(x, y) = \sum_{i=1}^{\infty} \tau_{a_i}(f_i(x), f_i(y))$, so

$$\begin{aligned} \rho(x,y) + \rho(y,z) &= \sum_{i=1}^{\infty} \tau_{a_i}(f_i(x), f_i(y)) + \sum_{i=1}^{\infty} \tau_{a_i}(f_i(y), f_i(z)) \\ &= \sum_{i=1}^{\infty} \{\tau_{a_i}(f_i(x), f_i(y)) + \tau_{a_i}(f_i(y), f_i(z))\} \qquad (2.7) \\ &\geqslant \sum_{i=1}^{\infty} \tau_{a_i}(f_i(x), f_i(z)) = \rho(x, z). \end{aligned}$$

Thus $\rho(\cdot, \cdot)$ is a metric over X.

Proposition 2.1 *Suppose $x_i \underset{\rho}{\to} x$. For any integer $N > 0$, there exists an integer $M > 0$ such that $f_n(x_m) = f_n(x)$ whenever $m \geqslant M$ and $1 \leqslant n \leqslant N$.*

Proof: Choose an integer $N > 0$. Since $x_i \underset{\rho}{\to} x$, there exists an integer $M > 0$ such that $\rho(x_m, x) < a_N$ whenever $m \geqslant M$. Now suppose $f_n(x_m) \neq f_n(x)$ for some $m \geqslant M$ and some n such that $1 \leqslant n \leqslant N$. Then $\rho(x_m, x) = \sum_{\substack{i=1 \\ f_i(x_m) \neq f_i(x)}}^{\infty} a_i \geqslant a_n \geqslant a_N > \rho(x_m, x)$ for a contradiction. It then follows that $f_n(x_m) = f_n(x)$ whenever $m \geqslant M$ and $1 \leqslant n \leqslant N$.

Proposition 2.2 *If $a \geqslant b$ then $\sigma(x, f_a(x)) \leqslant \sigma(x, r_b)$.*

Proof: From the definition, we earlier inferred that $\sigma(x, f_n(x)) = \min(\sigma(x, f_{n-1}(x)), \sigma(x, r_n))$. This implies $\sigma(x, f_n(x)) \leqslant \sigma(x, f_{n-1}(x))$, hence by induction $\sigma(x, f_a(x)) \leqslant \sigma(x, f_b(x))$. Next,

$\sigma(x, f_b(x)) = \min(\sigma(x, f_{b-1}(x)), \sigma(x, r_b)) \leqslant \sigma(x, r_b)$. Combining these results, we infer $\sigma(x, f_a(x)) \leqslant \sigma(x, f_b(x)) \leqslant \sigma(x, r_b)$, and the Proposition is proved.

Proposition 2.3 *If $x, y \in X$ and $x \neq y$, then given any integer $N > 0$, there exists $n > N$ such that $f_n(x) \neq f_n(y)$.*

Proof: Note that $f_n(x) \underset{\sigma}{\rightarrow} x$ and $f_n(y) \underset{\sigma}{\rightarrow} y$ as $n \rightarrow \infty$. Now suppose that for some $N > 0$, $f_n(x) = f_n(y)$ for all $n > N$. Then clearly $x = \lim_\sigma f_n(x) = \lim_\sigma f_n(y) = y$, thus $x = y$ for the contradiction.

Proposition 2.4 *For any $\epsilon > 0$ and $x \in X$, there exists an integer $N > 0$ such that if $f_j(x) = f_j(y)$ for some $j \geqslant N$ and some $y \in X$, then $\sigma(x, y) < \epsilon$.*

Proof: We prove the contraposition. Choose $\epsilon > 0$ and $x, y \in X$ such that $\sigma(x, y) \geqslant \epsilon$. Because (X, σ) is separable, we can find positive integers m, n such that $\sigma(x, r_m) < \frac{\epsilon}{2}$ and $\sigma(y, r_n) < \frac{\epsilon}{2}$. Let $N = \max(m, n)$. Suppose $f_j(x) = f_j(y)$ for some $j \geqslant N$. Then $\sigma(x, f_j(x)) \leqslant \sigma(x, r_m)$ and $\sigma(y, f_j(y)) \leqslant \sigma(y, r_n)$, and hence $\sigma(x, y) \leqslant \sigma(x, f_j(x)) + \sigma(f_j(x), f_j(y)) + \sigma(y, f_j(y)) < \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} = \epsilon$ for a contradiction. Hence $f_j(x) \neq f_j(y)$ for every $j \geqslant N$.

Proposition 2.5 *If $x_i \underset{\rho}{\rightarrow} x$, then $x_i \underset{\sigma}{\rightarrow} x$.*

Proof: Pick $\epsilon > 0$ and $x \in X$. Proposition 2.4 implies that there exists an integer $N > 0$ such that if $f_j(x) = f_j(y)$ for some $j \geqslant N$ and some $y \in X$, then $\sigma(x, y) < \epsilon$. Since $x_i \underset{\rho}{\rightarrow} x$, Proposition 2.1 implies that there exists an integer $M > 0$ such that $f_n(x_m) = f_n(x)$ whenever $m \geqslant M$ and $1 \leqslant n \leqslant N$. In particular, $f_N(x_m) = f_N(x)$ whenever $m \geqslant M$, hence $\sigma(x, x_m) < \epsilon$ whenever $m \geqslant M$, thus $x_i \underset{\sigma}{\rightarrow} x$.

Corollary 2.6 (X, ρ) *is no coarser than* (X, σ).

Remark Notice that Proposition 2.5 holds regardless of the choice of countable dense subset and the choice of non-increasing sequence of positive real numbers with convergent partial sums.

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