Statistical Magnitude of a Real Random Variable

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1 Introduction

Let X be a continuous real random variable. For any x_1 and x_2 such that $x_1 \leqslant x_2$, we define

$$\phi\left(x_1, x_2\right) \equiv \frac{P\left(x_2 \leqslant X\right)}{P\left(x_1 \leqslant X\right)}.$$

Note that

$$\phi\left(x_1, x_2\right) = \frac{P\left(x_2 \leqslant X \wedge x_1 \leqslant X\right)}{P\left(x_1 \leqslant X\right)} = P\left(x_2 \leqslant X | x_1 \leqslant X\right),$$

so $\phi\left(x_1, x_2\right)$ is seen to be the probability that X is at least x_2 given that X is at least the smaller value x_1. Note that we have the inequality $0 \leqslant \phi\left(x_1, x_2\right) \leqslant 1$. Note also that for $x_0 \leqslant x_1 \leqslant x_2$, we have $\phi\left(x_0, x_1\right) \phi\left(x_1, x_2\right) = \phi\left(x_0, x_2\right)$ We may regard $\phi\left(t_1, t_2\right)$ as the relative frequency of values not less than t_2 among a sample population consisting of values not less than t_1. We seek a real-valued function h such that

$$h(x_2) - h(x_1) = h(x_1) - h(x_0) \Leftrightarrow \phi\left(x_1, x_2\right) = \phi\left(x_0, x_1\right) \tag{1.1}$$

and call $h(x)$ a **statistical magnitude** of x.

2 Derivation

We now define the function

$$h\left(x\right) \equiv -\log P\left(x \leqslant X\right)$$

Note that

$$\begin{aligned} h\left(x_2\right) - h\left(x_1\right) &= -\log\left(P\left(x_2 \leqslant X\right)\right) + \log\left(P\left(x_1 \leqslant X\right)\right) \\ &= -\log\left(\frac{P\left(x_2 \leqslant X\right)}{P\left(x_1 \leqslant X\right)}\right) = -\log\left(\phi\left(x_1, x_2\right)\right) \end{aligned} \tag{2.1}$$

Now note that

$$\begin{aligned} h(x_2) - h(x_1) = h(x_1) - h(x_0) &\Leftrightarrow \frac{P(x_2 \leqslant X)}{P(x_1 \leqslant X)} = \frac{P(x_1 \leqslant X)}{P(x_0 \leqslant X)} \\ &\Leftrightarrow \phi\left(x_1, x_2\right) = \phi\left(x_0, x_1\right) \end{aligned} \tag{2.2}$$

Finally let us consider the distribution of the random variable

$$Y = h(X) \tag{2.3}$$

$$
\begin{aligned}
F_Y(y) = P(Y \leqslant y) = P(h(X) \leqslant y) = P\left(-\log\left(1 - F_X(X)\right) \leqslant y\right) \\
= P\left(1 - F_X(X) \geqslant \exp(-y)\right) = P\left(F_X(X) \leqslant 1 - \exp(-y)\right) \\
= P\left(X \leqslant F_X^{-1}\left(1 - \exp(-y)\right)\right) = F_X\left(F_X^{-1}\left(1 - \exp(-y)\right)\right) \\
= 1 - \exp(-y)
\end{aligned} \tag{2.4}
$$

so Y has the exponential distribution with parameter 1.

3 Examples

3.1 Example 1

Let X have the exponential distribution with parameter λ. Then

$$h(x) = -\log P(x \leqslant X) = -\log\exp(-\lambda x) = \lambda x$$

with

$$\phi(x_1, x_2) = \frac{P(x_2 \leqslant X)}{P(x_1 \leqslant X)} = \exp\left(-\lambda(x_2 - x_1)\right)$$

3.2 Example 2

Let X have the Rayleigh distribution with parameter σ, i.e.

$$f_X(x) = \frac{x e^{-\frac{x^2}{2\sigma^2}}}{\sigma^2}$$

for $x > 0$. then

$$h\left(x\right) = \frac{x^2}{2\sigma^2}$$

for $x > 0$.

3.3 Example 3

Let X have the uniform distribution over the interval $[a, b]$ with $a < b$.

Then

$$h\left(x\right) = \log\left(\frac{b-a}{b-x}\right)$$

for $a \leqslant x < b$.

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