State Vector Determination By a Single Tracking Satellite

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Abstract

Using only a single tracker capable only of range measurements to an orbiting object in an unknown Keplerian orbit, it is theoretically possible to calculate the orbit and a current state vector. In this article we present some equations relating time derivatives of the range squared to the orbital parameters of the tracked object.

Keywords. circular, distance, Keplerian, orbit, radius, range, satellite, tracking, vector, velocity.

1 Introduction

Suppose we have a satellite in circular orbit about some large spherical mass (henceforward called the **tracker**.) We assume the tracker is capable of measuring range and range rate to another satellite in an unknown orbit (henceforward called the **trackee**.). We denote the time-varying position vectors of the tracker and trackee by $\vec{R}$ and $\vec{r}$, respectively. To simplify the following derivation, we choose our time unit to be the time taken for the tracker to travel one radian in its orbit, and our distance unit to be the radius of the tracker's orbit. We may then write $\vec{R} = \hat{R}$ and $\vec{r} = r\,\hat{r}$ where $r = |\vec{r}\,|$.

In the following derivation, we will make use of the following equations:

$$\hat{R} \cdot \hat{T} = 0 \tag{1.1}$$

$$\frac{d\hat{R}}{dt} = \hat{T} \tag{1.2}$$

$$\frac{d\hat{T}}{dt} = -\hat{R} \tag{1.3}$$

$$\frac{d\vec{r}}{dt} = \vec{v} \tag{1.4}$$

$$\frac{d\vec{v}}{dt} = -\frac{1}{(\vec{r} \cdot \vec{r})^{\frac{3}{2}}}\vec{r} \tag{1.5}$$

$$r(t) \equiv \sqrt{\vec{r}(t) \cdot \vec{r}(t)} \tag{1.6}$$

where $\hat{T}$ is the velocity unit vector for the tracker.

2 Range Squared

We define $q(t) \equiv |\vec{r}(t) - \hat{R}(t)|^2$. Equivalently, we have

$$q(t) = \left(\vec{r}(t) - \hat{R}(t)\right) \cdot \left(\vec{r}(t) - \hat{R}(t)\right) = \vec{r}(t) \cdot \vec{r}(t) - 2\,\vec{r}(t) \cdot \hat{R}(t) + \hat{R}(t) \cdot \hat{R}(t)$$
$$= \vec{r}(t) \cdot \vec{r}(t) - 2\,\vec{r}(t) \cdot \hat{R}(t) + 1 = r(t)^2 - 2\,\vec{r}(t) \cdot \hat{R}(t) + 1 \tag{2.1}$$

Taking time derivatives of q and making substitutions yields

$$q'(t) = 2\vec{r}(t) \cdot \vec{v}(t) - 2(\vec{v}(t) \cdot \hat{R}(t) + \vec{r}(t) \cdot \hat{T}(t)) \tag{2.2}$$

$$\begin{aligned} q''(t) = 2\left(\vec{v}(t) \cdot \vec{v}(t) - \frac{1}{\sqrt{\vec{r}(t) \cdot \vec{r}(t)}}\right) \\ -2\left(-\frac{\vec{r}(t) \cdot \hat{R}(t)}{(\vec{r}(t) \cdot \vec{r}(t))^{\frac{3}{2}}} - \vec{r}(t) \cdot \hat{R}(t) + 2\vec{v}(t) \cdot \hat{T}(t)\right) \end{aligned} \tag{2.3}$$

$$\begin{aligned} q'''(t) = -\frac{6\vec{r}(t) \cdot \hat{R}(t)\vec{r}(t) \cdot \vec{v}(t)}{(\vec{r}(t) \cdot \vec{r}(t))^{\frac{5}{2}}} + \frac{2\vec{v}(t) \cdot \hat{R}(t)}{(\vec{r}(t) \cdot \vec{r}(t))^{\frac{3}{2}}} + 6\vec{v}(t) \cdot \hat{R}(t) \\ + \frac{6\vec{r}(t) \cdot \hat{T}(t)}{(\vec{r}(t) \cdot \vec{r}(t))^{\frac{3}{2}}} + 2\vec{r}(t) \cdot \hat{T}(t) - \frac{2\vec{r}(t) \cdot \vec{v}(t)}{(\vec{r}(t) \cdot \vec{r}(t))^{\frac{3}{2}}} \end{aligned} \tag{2.4}$$

$$\begin{aligned}
q^{(4)}(t) = {} & \frac{144\vec{r}(t)\cdot\hat{R}(t)\vec{r}(t)\cdot\hat{T}(t)\vec{r}(t)\cdot\vec{v}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^4} \\
& + \frac{48\vec{r}(t)\cdot\hat{R}(t)\vec{r}(t)\cdot\hat{T}(t)\vec{r}(t)\cdot\vec{v}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{11/2}} - \frac{48\vec{r}(t)\cdot\hat{R}(t)\vec{v}(t)\cdot\hat{T}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{3/2}} \\
& - \frac{64\vec{r}(t)\cdot\hat{R}(t)\vec{v}(t)\cdot\hat{T}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{3}} - \frac{16\vec{r}(t)\cdot\hat{R}(t)\vec{v}(t)\cdot\hat{T}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{9/2}} \\
& - \frac{180(\vec{r}(t)\cdot\hat{R}(t))^2(\vec{r}(t)\cdot\vec{v}(t))^2}{(\vec{r}(t)\cdot\vec{r}(t))^{5}} - \frac{60(\vec{r}(t)\cdot\hat{R}(t))^2(\vec{r}(t)\cdot\vec{v}(t))^2}{(\vec{r}(t)\cdot\vec{r}(t))^{13/2}} \\
& + \frac{36\vec{v}(t)\cdot\vec{v}(t)(\vec{r}(t)\cdot\hat{R}(t))^2}{(\vec{r}(t)\cdot\vec{r}(t))^{4}} + \frac{12\vec{v}(t)\cdot\vec{v}(t)(\vec{r}(t)\cdot\hat{R}(t))^2}{(\vec{r}(t)\cdot\vec{r}(t))^{11/2}} \\
& + \frac{30\vec{r}(t)\cdot\hat{R}(t)(\vec{r}(t)\cdot\vec{v}(t))^2}{(\vec{r}(t)\cdot\vec{r}(t))^{7/2}} - \frac{36\vec{r}(t)\cdot\hat{R}(t)(\vec{r}(t)\cdot\vec{v}(t))^2}{(\vec{r}(t)\cdot\vec{r}(t))^{4}} \\
& - \frac{12\vec{r}(t)\cdot\hat{R}(t)(\vec{r}(t)\cdot\vec{v}(t))^2}{(\vec{r}(t)\cdot\vec{r}(t))^{11/2}} + \frac{72\vec{r}(t)\cdot\hat{R}(t)\vec{v}(t)\cdot\hat{R}(t)\vec{r}(t)\cdot\vec{v}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{4}} \\
& + \frac{24\vec{r}(t)\cdot\hat{R}(t)\vec{v}(t)\cdot\hat{R}(t)\vec{r}(t)\cdot\vec{v}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{11/2}} - \frac{6\vec{v}(t)\cdot\vec{v}(t)\vec{r}(t)\cdot\hat{R}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{5/2}} \\
& + \frac{12\vec{v}(t)\cdot\vec{v}(t)\vec{r}(t)\cdot\hat{R}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{3}} + \frac{4\vec{v}(t)\cdot\vec{v}(t)\vec{r}(t)\cdot\hat{R}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{9/2}} \\
& - \frac{12\vec{v}(t)\cdot\hat{R}(t)\vec{r}(t)\cdot\vec{v}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{5/2}} + \frac{12(\vec{r}(t)\cdot\hat{R}(t))^2}{(\vec{r}(t)\cdot\vec{r}(t))^{3/2}} + \frac{76(\vec{r}(t)\cdot\hat{R}(t))^2}{(\vec{r}(t)\cdot\vec{r}(t))^{3}} \\
& - \frac{8(\vec{r}(t)\cdot\hat{R}(t))^2}{(\vec{r}(t)\cdot\vec{r}(t))^{6}} - \frac{6\vec{r}(t)\cdot\hat{R}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{3/2}} + \frac{6\vec{r}(t)\cdot\hat{R}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{3}} - \frac{12\vec{r}(t)\cdot\hat{R}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{7/2}} \\
& - \frac{4\vec{r}(t)\cdot\hat{R}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{5}} - 2\vec{r}(t)\cdot\hat{R}(t) - \frac{24\vec{r}(t)\cdot\hat{T}(t)\vec{r}(t)\cdot\vec{v}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{5/2}} \\
& + \frac{8\vec{v}(t)\cdot\hat{T}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{3/2}} + \frac{6(\vec{r}(t)\cdot\vec{v}(t))^2}{(\vec{r}(t)\cdot\vec{r}(t))^{5/2}} - \frac{2\vec{v}(t)\cdot\vec{v}(t)}{(\vec{r}(t)\cdot\vec{r}(t))^{3/2}} \\
& + \frac{2}{(\vec{r}(t)\cdot\vec{r}(t))^{2}} + 8\vec{v}(t)\cdot\hat{T}(t)
\end{aligned} \tag{2.5}$$

Now suppose at some instant t_0 we measure q and the first through the third time derivatives of q at the same instant. This could be accomplished, for instance, through numerical methods using several measurements about

t_0, resulting in the numbers m_0, m_1, m_2 and m_3 as follows:

$$\begin{aligned} m_0 &= q(t_0) \\ m_1 &= q'(t_0) \\ m_2 &= q''(t_0) \\ m_3 &= q'''(t_0) \\ m_4 &= q^{(4)}(t_0) \end{aligned} \tag{2.6}$$

2.1 The Angle Between Orbital Planes

Let ι represent the angle subtended by the orbital planes of the tracker and trackee. Define

$$\vec{h} = \vec{r} \times \vec{v}. \tag{2.7}$$

Then

$$\cos\iota = \hat{h} \cdot \hat{H} \tag{2.8}$$

$$\cos\iota = \frac{r_r\, v_t - r_t\, v_r}{\sqrt{\alpha_1}} \tag{2.9}$$

or

$$\cos\iota = \frac{r_r\, v_t - r_t\, v_r}{\sqrt{\alpha_2}} \tag{2.10}$$

i.e.

$$\iota = \arccos\left(\frac{r_r\, v_t - r_t\, v_r}{\sqrt{\alpha_1}}\right) \tag{2.11}$$

or

$$\iota = \arccos\left(\frac{r_r\, v_t - r_t\, v_r}{\sqrt{\alpha_2}}\right) \tag{2.12}$$

where

$$\begin{aligned} r_r &= \vec{r}\cdot\hat{R} \\ r_t &= \vec{r}\cdot\hat{T} \\ v_r &= \vec{v}\cdot\hat{R} \\ v_t &= \vec{v}\cdot\hat{T} \\ r &= \sqrt{\vec{r}\cdot\vec{r}} \\ v &= \sqrt{\vec{v}\cdot\vec{v}}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} \alpha_1 = 2r_r\, v_r &\left(\sqrt{r^2 - r_r^2 - r_t^2}\sqrt{v^2 - v_r^2 - v_t^2} - r_t\, v_t\right) \\ +2r_t\, v_t\sqrt{r^2 - r_r^2 - r_t^2}&\sqrt{v^2 - v_r^2 - v_t^2} + r^2\left(v_r^2 + v_t^2\right) \\ &+r_r^2\left(v^2 - 2v_r^2 - v_t^2\right) + r_t^2\left(v^2 - v_r^2 - 2v_t^2\right) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \alpha_2 = -2r_r\, v_r &\left(\sqrt{r^2 - r_r^2 - r_t^2}\sqrt{v^2 - v_r^2 - v_t^2} + r_t\, v_t\right) \\ -2r_t\, v_t\sqrt{r^2 - r_r^2 - r_t^2}&\sqrt{v^2 - v_r^2 - v_t^2} + r^2\left(v_r^2 + v_t^2\right) \\ &+r_r^2\left(v^2 - 2v_r^2 - v_t^2\right) + r_t^2\left(v^2 - v_r^2 - 2v_t^2\right) \end{aligned} \tag{2.15}$$

3 Cases

3.1 Circular Orbit

In this section we assume the tracked object is in a circular orbit, i.e. $\vec{r}(t_0) \cdot \vec{v}(t_0) = 0\cdot$

3.1.1 Radius $= 1$

In this subsection we assume that $\vec{r}(t_0) \cdot \vec{r}(t_0) = 1\cdot$

Under these assumptions, we have

$$\begin{aligned} 2 - 2\vec{r}(t_0) \cdot \hat{R}(t_0) &= m_0 \\ -2(\vec{v}(t_0) \cdot \hat{R}(t_0) + \vec{r}(t_0) \cdot \hat{T}(t_0)) &= m_1 \\ 2(2\vec{r}(t_0) \cdot \hat{R}(t_0) - 2\vec{v}(t_0) \cdot \hat{T}(t_0)) &= m_2 \end{aligned} \tag{3.1}$$

Solving simultaneously, we get

$$\begin{aligned} \vec{r}(t_0) \cdot \hat{R}(t_0) &= \tfrac{2-m_0}{2} \\ \vec{v}(t_0) \cdot \hat{T}(t_0) &= \tfrac{1}{4}(-2m_0 - m_2 + 4) \\ \vec{r}(t_0) \cdot \hat{T}(t_0) &= \tfrac{1}{2}(-m_1 - 2\vec{v}(t_0) \cdot \hat{R}(t_0)) \end{aligned} \tag{3.2}$$

$$\hat{r}(t_0) \cdot \hat{H}(t_0) = -\sqrt{1 - \frac{1}{4}(2 - m_0)^2 - \frac{1}{4}(-m_1 - 2\hat{v}(t_0) \cdot \hat{R}(t_0))^2} \tag{3.3}$$

or

$$\hat{r}(t_0) \cdot \hat{H}(t_0) = \sqrt{1 - \frac{1}{4}(2 - m_0)^2 - \frac{1}{4}(-m_1 - 2\hat{v}(t_0) \cdot \hat{R}(t_0))^2} \tag{3.4}$$

$$\hat{r} = \hat{r}(t_0) \cdot \hat{R}(t_0)\hat{R}(t_0) + \hat{r}(t_0) \cdot \hat{T}(t_0)\hat{T}(t_0) + \hat{r}(t_0) \cdot \hat{H}(t_0)\hat{H}(t_0) \tag{3.5}$$

$$4q_1(t) + q_3(t) \equiv 0 \tag{3.6}$$

$$\begin{aligned} \hat{r} = \hat{H}(t_0) &\left(-\sqrt{-\frac{1}{4}(m_0 - 2)^2 - \frac{1}{4}(m_1 + 2\vec{v}(t_0) \cdot \hat{R}(t_0))^2 + 1} \right) \\ &-\frac{1}{2}m_0\hat{R}(t_0) - \frac{1}{2}m_1\hat{T}(t_0) - \hat{T}(t_0)\vec{v}(t_0) \cdot \hat{R}(t_0) + \hat{R}(t_0) \end{aligned} \tag{3.7}$$

or

$$\begin{aligned} \hat{r} = \hat{H}(t_0)&\sqrt{-\frac{1}{4}(m_0 - 2)^2 - \frac{1}{4}(m_1 + 2\vec{v}(t_0) \cdot \hat{R}(t_0))^2 + 1} \\ &-\frac{1}{2}m_0\hat{R}(t_0) - \frac{1}{2}m_1\hat{T}(t_0) - \hat{T}(t_0)\vec{v}(t_0) \cdot \hat{R}(t_0) + \hat{R}(t_0) \end{aligned} \tag{3.8}$$

$$\begin{aligned} \hat{v} = \hat{H}(t_0) &\left(-\sqrt{-\frac{1}{16}(2m_0 + m_2 - 4)^2 - (\hat{v}(t_0) \cdot \hat{R}(t_0))^2 + 1} \right) \\ &-\frac{1}{2}m_0\hat{T}(t_0) - \frac{1}{4}m_2\hat{T}(t_0) + \hat{R}(t_0)\hat{v}(t_0) \cdot \hat{R}(t_0) + \hat{T}(t_0) \end{aligned} \tag{3.9}$$

or

$$\hat{v} = \hat{H}(t_0)\sqrt{-\frac{1}{16}(2m_0 + m_2 - 4)^2 - (\hat{v}(t_0)\cdot\hat{R}(t_0))^2 + 1} \\ -\frac{1}{2}m_0\hat{T}(t_0) - \frac{1}{4}m_2\hat{T}(t_0) + \hat{R}(t_0)\hat{v}(t_0)\cdot\hat{R}(t_0) + \hat{T}(t_0) \tag{3.10}$$

The planar angle ι is then either

$$\frac{\text{num}}{\sqrt{\text{den1s}}} \tag{3.11}$$

or

$$\frac{\text{num}}{\sqrt{\text{den2s}}} \tag{3.12}$$

where

$$num = 2m_0^2 + m_0m_2 - 8m_0 + 4m_1\vec{r}(t)\cdot\hat{T}(t) - 2m_2 + 8(\vec{r}(t)\cdot\hat{T}(t))^2 + 8 \tag{3.13}$$

$$
\begin{aligned}
den1s = & -4\sqrt{den1s1}\sqrt{den1s2}m_0m_1 + 8\sqrt{den1s1}\sqrt{den1s2}m_1 \\
& +4\sqrt{den1s1}\sqrt{den1s2}m_2\vec{r}(t)\cdot\hat{T}(t) - 4m_0^4 - 4m_0^3m_2 + 32m_0^3 \\
& -8m_0^2m_1^2 - 16m_0^2m_1\vec{r}(t)\cdot\hat{T}(t) - m_0^2m_2^2 + 24m_0^2m_2 \\
& -32m_0^2(\vec{r}(t)\cdot\hat{T}(t))^2 - 64m_0^2 + 32m_0m_1^2 + 8m_0m_1m_2\vec{r}(t)\cdot\hat{T}(t) \\
& +64m_0m_1\vec{r}(t)\cdot\hat{T}(t) + 4m_0m_2^2 - 16m_0m_2(\vec{r}(t)\cdot\hat{T}(t))^2 - 32m_0m_2 \\
& +128m_0(\vec{r}(t)\cdot\hat{T}(t))^2 - 16m_1^2(\vec{r}(t)\cdot\hat{T}(t))^2 - 16m_1^2 \\
& -16m_1m_2\vec{r}(t)\cdot\hat{T}(t) - 64m_1(\vec{r}(t)\cdot\hat{T}(t))^3 - 8m_2^2(\vec{r}(t)\cdot\hat{T}(t))^2 \\
& +32m_2(\vec{r}(t)\cdot\hat{T}(t))^2 - 64(\vec{r}(t)\cdot\hat{T}(t))^4 + 64
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
den1s1 = & -4m_0^2 - 4m_0(m_2 - 4) - 4m_1^2 - 16m_1\vec{r}(t)\cdot Tu(t) - m_2^2 \\
& +8m_2 - 16(\vec{r}(t)\cdot Tu(t))^2
\end{aligned} \tag{3.15}
$$

$$
den1s2 = (4 - m_0)m_0 - 4(rv(t)\cdot\hat{T}(t))^2 \tag{3.16}
$$

$$
\begin{aligned}
den2s = {} & 4\sqrt{den2s1}\sqrt{den2s2}\, m_0 m_1 - 8\sqrt{den2s1}\sqrt{den2s2}\, m_1 \\
& -4\sqrt{den2s1}\sqrt{den2s2}\, m_2 \vec{r}(t)\cdot\hat{T}(t) - 4m_0^4 - 4m_0^3 m_2 + 32m_0^3 \\
& -8m_0^2 m_1^2 - 16m_0^2 m_1 \vec{r}(t)\cdot\hat{T}(t) - m_0^2 m_2^2 + 24m_0^2 m_2 \\
& -32m_0^2(\vec{r}(t)\cdot\hat{T}(t))^2 - 64m_0^2 + 32m_0 m_1^2 + 8m_0 m_1 m_2 \vec{r}(t)\cdot\hat{T}(t) \\
& +64m_0 m_1 \vec{r}(t)\cdot\hat{T}(t) + 4m_0 m_2^2 - 16m_0 m_2(\vec{r}(t)\cdot\hat{T}(t))^2 - 32m_0 m_2 \\
& +128m_0(\vec{r}(t)\cdot\hat{T}(t))^2 - 16m_1^2(\vec{r}(t)\cdot\hat{T}(t))^2 - 16m_1^2 \\
& -16m_1 m_2 \vec{r}(t)\cdot\hat{T}(t) - 64m_1(\vec{r}(t)\cdot\hat{T}(t))^3 - 8m_2^2(\vec{r}(t)\cdot\hat{T}(t))^2 \\
& +32m_2(\vec{r}(t)\cdot\hat{T}(t))^2 - 64(\vec{r}(t)\cdot\hat{T}(t))^4 + 64
\end{aligned} \tag{3.17}
$$

$$
\begin{aligned}
den2s1 = {} & -4m_0^2 - 4m_0 m_2 + 16m_0 - 4m_1^2 - 16m_1 \vec{r}(t)\cdot\hat{T}(t) - m_2^2 \\
& +8m_2 - 16(\vec{r}(t)\cdot\hat{T}(t))^2
\end{aligned} \tag{3.18}
$$

$$
den2s2 = -m_0^2 + 4m_0 - 4(\vec{r}(t)\cdot\hat{T}(t))^2 \tag{3.19}
$$

3.1.2 Radius $\neq 1$

Under this assumption, we have

$$
\begin{aligned}
-2\vec{r}(t_0)\cdot\hat{R}(t_0)+r^2+1 &= m_0\\
-2(\vec{v}(t_0)\cdot\hat{R}(t_0)+\vec{r}(t_0)\cdot\hat{T}(t_0)) &= m_1\\
2\left(\left(\tfrac{1}{r^3}+1\right)\vec{r}(t_0)\cdot\hat{R}(t_0)-2\vec{v}(t_0)\cdot\hat{T}(t_0)\right) &= m_2\\
\left(\tfrac{2}{r^3}+6\right)\vec{v}(t_0)\cdot\hat{R}(t_0)+\left(\tfrac{6}{r^3}+2\right)\vec{r}(t_0)\cdot\hat{T}(t_0) &= m_3
\end{aligned}
\tag{3.20}
$$

Solving simultaneously, we get

$$
\begin{aligned}
\vec{r}(t_0)\cdot\hat{R}(t_0) &= \tfrac{1}{2}\left(-m_0+r^2+1\right)\\
\vec{v}(t_0)\cdot\hat{R}(t_0) &= \tfrac{m_1\left(r^3+3\right)+m_3r^3}{4(r^3-1)}\\
\vec{r}(t_0)\cdot\hat{T}(t_0) &= \tfrac{3m_1r^3+m_1+m_3r^3}{4-4r^3}\\
\vec{v}(t_0)\cdot\hat{T}(t_0) &= \tfrac{1}{4}(-m_0-m_2+1)+\tfrac{1-m_0}{4r^3}+\tfrac{r^2}{4}-\tfrac{1}{4r}+\tfrac{v^2}{2}
\end{aligned}
\tag{3.21}
$$

Solving

$$
r^2=(\vec{r}(t_0)\cdot\hat{H}(t_0))^2+(\vec{r}(t_0)\cdot\hat{R}(t_0))^2+(\vec{r}(t_0)\cdot\hat{T}(t))^2 \tag{3.22}
$$

we get

$$
\vec{r}(t_0)\cdot\hat{H}(t_0)=-\frac{\sqrt{A}}{4\left|r^3-1\right|} \tag{3.23}
$$

or

$$
\vec{r}(t_0)\cdot\hat{H}(t_0)=\frac{\sqrt{A}}{4\left|r^3-1\right|} \tag{3.24}
$$

where

$$A \equiv -4m_0^2\left(r^3-1\right)^2+16m_0\left(r^3-1\right)^2-\left(3m_1r^3+m_1+m_3r^3\right)^2 \tag{3.25}$$

Solving

$$\frac{1}{r}=(\vec{v}(t_0)\cdot\hat{H}(t_0))^2+(\vec{v}(t_0)\cdot\hat{R}(t_0))^2+(\vec{v}(t_0)\cdot\hat{T}(t_0))^2 \tag{3.26}$$

we get

$$\vec{v}(t_0)\cdot\hat{H}(t_0)=-\frac{\sqrt{B}}{4\left|r^3-1\right|} \tag{3.27}$$

or

$$\vec{v}(t_0)\cdot\hat{H}(t_0)=\frac{\sqrt{B}}{4\left|r^3-1\right|} \tag{3.28}$$

where

$$\begin{aligned} B \equiv r^6&\left(-4m_0^2+(8-4m_0)m_2+16m_0-m_1^2-2m_1m_3-m_2^2-m_3^2\right)\\ &+r^3\left(8m_0^2+(8m_0-16)m_2-32m_0-6m_1^2-6m_1m_3+2m_2^2\right)\\ &-4m_0^2+(8-4m_0)m_2+16m_0-9m_1^2-m_2^2 \end{aligned} \tag{3.29}$$

$$\begin{aligned} \vec{r}&=\frac{1}{2}\left(1-m_0+r^2\right)\hat{R}(t_0)\\ -\frac{3m_1r^3+m_1+m_3r^3}{4\left(r^3-1\right)}&\hat{T}(t_0)+\vec{r}(t_0)\cdot\hat{H}(t_0)\hat{H}(t_0) \end{aligned} \tag{3.30}$$

$$
\begin{aligned}
\vec{v} = \frac{\left(m_3 r^3 + m_1\left(r^3+3\right)\right)}{4\left(r^3-1\right)}\hat{R}(t_0) \\
+\frac{1}{4}\left(r^2 - m_2 + m_0\left(-1-\frac{1}{r^3}\right)+1+\frac{1}{r}+\frac{1}{r^3}\right)\hat{T}(t_0) \\
-\frac{\sqrt{C_1}}{4\left|r^3-1\right|}\hat{H}(t_0)
\end{aligned}
\tag{3.31}
$$

or

$$
\begin{aligned}
\vec{v} = \frac{\left(m_3 r^3 + m_1\left(r^3+3\right)\right)}{4\left(r^3-1\right)}\hat{R}(t_0) \\
+\frac{1}{4}\left(r^2 - m_2 + m_0\left(-1-\frac{1}{r^3}\right)+1+\frac{1}{r}+\frac{1}{r^3}\right)\hat{T}(t_0) \\
+\frac{\sqrt{C_1}}{4\left|r^3-1\right|}\hat{H}(t_0)
\end{aligned}
\tag{3.32}
$$

where

$$
\begin{aligned}
C_1 = -m_2^2 r^6 - m_3^2 r^6 + 8m_2 r^6 + 2m_2^2 r^3 - 16 m_2 r^3 \\
-2m_1 m_3\left(r^3+3\right)r^3 - m_2^2 - 4m_0^2\left(r^3-1\right)^2 \\
-4m_0(m_2-4)\left(r^3-1\right)^2 - m_1^2\left(r^3+3\right)^2 + 8m_2
\end{aligned}
\tag{3.33}
$$

3.2 Noncircular Orbit

In this subsection we assume the tracked object is in a noncircular orbit. In this case we have

$$
\begin{aligned}
-2\vec{r}(t_0)\cdot\hat{R}(t_0)+\vec{r}(t_0)\cdot\vec{r}(t_0)+1=m_0\\
-2(\vec{v}(t_0)\cdot\hat{R}(t_0)+\vec{r}(t_0)\cdot\hat{T}(t_0)-\vec{r}(t_0)\cdot\vec{v}(t_0))=m_1\\
\tfrac{2\vec{r}(t_0)\cdot\hat{R}(t_0)}{(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}+2\vec{r}(t_0)\cdot\hat{R}(t_0)-\tfrac{2}{\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}\\
-4\vec{v}(t_0)\cdot\hat{T}(t_0)+2\vec{v}(t_0)\cdot\vec{v}(t_0)=m_2\\
-\tfrac{6\vec{r}(t_0)\cdot\hat{R}(t_0)\vec{r}(t_0)\cdot\vec{v}(t_0)}{(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{5}{2}}}+\tfrac{2\vec{v}(t_0)\cdot\hat{R}(t_0)}{(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}+6\vec{v}(t_0)\cdot\hat{R}(t_0)\\
+\tfrac{6\vec{r}(t_0)\cdot\hat{T}(t_0)}{(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}+2\vec{r}(t_0)\cdot\hat{T}(t_0)-\tfrac{2\vec{r}(t_0)\cdot\vec{v}(t_0)}{(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}=m_3
\end{aligned}
\tag{3.34}
$$

Solving simultaneously, we get

$$
\begin{aligned}
\vec{r}(t_0)\cdot\hat{R}(t_0)=\tfrac{1}{2}(-m_0+\vec{r}(t_0)\cdot\vec{r}(t_0)+1)\\
\vec{v}(t_0)\cdot\hat{R}(t_0)\\
=\tfrac{-3(m_0-1)\vec{r}(t_0)\cdot\vec{v}(t_0)+(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{5}{2}}(m_1+m_3-2\vec{r}(t_0)\cdot\vec{v}(t_0))+\vec{r}(t_0)\cdot\vec{r}(t_0)(3m_1-\vec{r}(t_0)\cdot\vec{v}(t_0))}{4\vec{r}(t_0)\cdot\vec{r}(t_0)\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)}\\
\vec{r}(t_0)\cdot\hat{T}(t_0)\\
=-\tfrac{-3(m_0-1)\vec{r}(t_0)\cdot\vec{v}(t_0)+(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{5}{2}}(3m_1+m_3-6\vec{r}(t_0)\cdot\vec{v}(t_0))+\vec{r}(t_0)\cdot\vec{r}(t_0)(m_1+3\vec{r}(t_0)\cdot\vec{v}(t_0))}{4\vec{r}(t_0)\cdot\vec{r}(t_0)\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)}\\
\vec{v}(t_0)\cdot\hat{T}(t_0)\\
=\tfrac{-(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}(m_0+m_2-2\vec{v}(t_0)\cdot\vec{v}(t_0)-1)-m_0+(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{5}{2}}-\vec{r}(t_0)\cdot\vec{r}(t_0)+1}{4(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}
\end{aligned}
\tag{3.35}
$$

Solving

$$
\vec{r}(t_0)\cdot\vec{r}(t_0)=(\vec{r}(t_0)\cdot\hat{H}(t_0))^2+(\vec{r}(t_0)\cdot\hat{R}(t_0))^2+(\vec{r}(t_0)\cdot\hat{T}(t))^2 \tag{3.36}
$$

we get

$$\vec{r}(t_0) \cdot \hat{H}(t_0) = -\sqrt{C_2 + C_3} \tag{3.37}$$

or

$$\vec{r}(t_0) \cdot \hat{H}(t_0) = \sqrt{C_2 + C_3} \tag{3.38}$$

where

$$\begin{aligned} C_2 = &-\frac{9(\vec{r}(t_0)\cdot\vec{v}(t_0))^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2} \\ &+\frac{9m_1\vec{r}(t_0)\cdot\vec{v}(t_0)(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}+\frac{3m_3\vec{r}(t_0)\cdot\vec{v}(t_0)(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2} \\ &-\frac{9m_1^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}{16\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}-\frac{m_3^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}{16\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2} \\ &-\frac{3m_1m_3(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}-\frac{1}{4}(\vec{r}(t_0)\cdot\vec{r}(t_0))^2 \\ &+\frac{9(\vec{r}(t_0)\cdot\vec{v}(t_0))^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}-\frac{3m_1\vec{r}(t_0)\cdot\vec{v}(t_0)(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2} \\ &-\frac{3m_3\vec{r}(t_0)\cdot\vec{v}(t_0)(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}-\frac{3m_1^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2} \\ &-\frac{m_1m_3(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}+\frac{1}{2}m_0\vec{r}(t_0)\cdot\vec{r}(t_0) \\ &+\frac{1}{2}\vec{r}(t_0)\cdot\vec{r}(t_0)-\frac{9m_0(\vec{r}(t_0)\cdot\vec{v}(t_0))^2\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2} \end{aligned} \tag{3.39}$$

$$
\begin{aligned}
C_3 = {} & \frac{9(\vec{r}(t_0)\cdot\vec{v}(t_0))^2\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2} \\
& + \frac{9m_0m_1\vec{r}(t_0)\cdot\vec{v}(t_0)\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2} - \frac{9m_1\vec{r}(t_0)\cdot\vec{v}(t_0)\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2} \\
& + \frac{3m_0m_3\vec{r}(t_0)\cdot\vec{v}(t_0)\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2} - \frac{3m_3\vec{r}(t_0)\cdot\vec{v}(t_0)\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2} \\
& - \frac{m_0^2}{4} - \frac{9(\vec{r}(t_0)\cdot\vec{v}(t_0))^2}{16\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2} + \frac{m_0}{2} - \frac{3m_1\vec{r}(t_0)\cdot\vec{v}(t_0)}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2} \\
& - \frac{m_1^2}{16\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2} - \frac{1}{4} + \frac{9m_0(\vec{r}(t_0)\cdot\vec{v}(t_0))^2}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2\vec{r}(t_0)\cdot\vec{r}(t_0)} \\
& - \frac{9(\vec{r}(t_0)\cdot\vec{v}(t_0))^2}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2\vec{r}(t_0)\cdot\vec{r}(t_0)} + \frac{3m_0m_1\vec{r}(t_0)\cdot\vec{v}(t_0)}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2\vec{r}(t_0)\cdot\vec{r}(t_0)} \\
& - \frac{3m_1\vec{r}(t_0)\cdot\vec{v}(t_0)}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2\vec{r}(t_0)\cdot\vec{r}(t_0)} - \frac{9m_0^2(\vec{r}(t_0)\cdot\vec{v}(t_0))^2}{16\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^2} \\
& + \frac{9m_0(\vec{r}(t_0)\cdot\vec{v}(t_0))^2}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^2} - \frac{9(\vec{r}(t_0)\cdot\vec{v}(t_0))^2}{16\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^2}
\end{aligned}
\tag{3.40}
$$

This implies

$$
\vec{r}\cdot\vec{v} = \frac{N1s}{denom} \tag{3.41}
$$

where

$$
\begin{aligned}
N1s = \pm\sqrt{N1s0 + N1s2\, v^2} \\
+3r^2\left(3m_0r^3 + m_0 + r^6 - 3r^5 - 3r^3 - r^2 - 1\right)\left(m_1\left(5r^3 - 1\right) + m_3r^3\right) \\
N1s0 = N1s000r^0 + N1s001r^1 + N1s002r^2 + N1s003r^3 \\
+N1s004r^4 + N1s005r^5 + N1s006r^6 + N1s007r^7 + N1s008r^8 \\
+N1s009r^9 + N1s010r^{10} + N1s011r^{11} + N1s012r^{12} + N1s013r^{13} \\
+N1s014r^{14} + N1s015r^{15} + N1s016r^{16} + N1s017r^{17} + N1s018r^{18} \\
+N1s019r^{19} + N1s020r^{20} + N1s021r^{21} + N1s022r^{22} + N1s023r^{23} \\
+N1s024r^{24} + N1s025r^{25} + N1s026r^{26} + N1s027r^{27} + N1s028r^{28} \\
+N1s029r^{29} + N1s030r^{30} + N1s031r^{31} + N1s032r^{32} + N1s033r^{33} \\
+N1s034r^{34} + N1s035r^{35} + N1s036r^{36} + N1s037r^{37} + N1s038r^{38} \\
+N1s039r^{39} + N1s040r^{40}
\end{aligned}
\tag{3.42}
$$

$$
\begin{aligned}
N1s000 &= 0 \\
N1s001 &= -96m_0^4 + 384m_0^3 - 576m_0^2 + 384m_0 - 96 \\
N1s002 &= 0 \\
N1s003 &= 240m_0^3 - 720m_0^2 + 720m_0 - 240 \\
N1s004 &= -192m_0^4 - 48m_2m_0^3 + 768m_0^3 + 9m_1^2m_0^2 \\
&+144m_2m_0^2 - 1152m_0^2 - 18m_1^2m_0 - 144m_2m_0 \\
&+768m_0 + 9m_1^2 + 48m_2 - 192 \\
N1s005 &= -192m_0^2 + 384m_0 - 192 \\
N1s006 &= 504m_0^3 + 96m_2m_0^2 - 1512m_0^2 - 18m_1^2m_0 \\
&-192m_2m_0 + 1512m_0 + 18m_1^2 + 96m_2 - 504 \\
N1s007 &= 1008m_0^4 - 168m_2m_0^3 - 4248m_0^3 \\
&-36m_1^2m_0^2 + 504m_2m_0^2 - 18m_1m_3m_0^2 + 6696m_0^2 \\
&+72m_1^2m_0 - 504m_2m_0 \\
N1s008 &= +36m_1m_3m_0 - 4632m_0 \\
&-36m_1^2 + 168m_2 - 18m_1m_3 + 1176
\end{aligned}
\tag{3.43}
$$

$$
\begin{aligned}
N1s009 &= -504m_0^2 - 48m_2m_0 + 1008m_0 + 9m_1^2 + 48m_2 - 504 \\
N1s010 &= -2160m_0^3 + 408m_2m_0^2 + 6816m_0^2 + 72m_1^2m_0 \\
&-816m_2m_0 + 36m_1m_3m_0 - 7152m_0 - 72m_1^2 + 408m_2 \\
&-36m_1m_3 + 2496 \\
N1s011 &= 1068m_0^4 + 336m_2m_0^3 - 4140m_0^3 - 234m_1^2m_0^2 \\
&+9m_3^2m_0^2 - 1104m_2m_0^2 - 18m_1m_3m_0^2 + 6012m_0^2 \\
&+486m_1^2m_0 - 18m_3^2m_0 \\
N1s012 &= +1200m_2m_0 + 36m_1m_3m_0 - 3612m_0 - 252m_1^2 \\
&+9m_3^2 - 432m_2 - 18m_1m_3 + 672 \\
N1s013 &= 1152m_0^2 - 312m_2m_0 - 2424m_0 - 36m_1^2 + 312m_2 \\
&-18m_1m_3 + 1272 \\
N1s014 &= -1392m_0^3 - 240m_2m_0^2 + 4092m_0^2 + 468m_1^2m_0 \\
&-18m_3^2m_0 + 576m_2m_0 + 36m_1m_3m_0 \\
N1s015 &= -4008m_0 - 486m_1^2 + 18m_3^2 - 336m_2 - 36m_1m_3 + 1236 \\
N1s016 &= -2724m_0^4 + 1248m_2m_0^3 + 12564m_0^3 + 540m_1^2m_0^2 \\
&+54m_3^2m_0^2 - 3792m_2m_0^2
\end{aligned}
\tag{3.44}
$$

$$
\begin{aligned}
N1s017 &= +378m_1m_3m_0^2 - 24m_4m_0^2 - 21468m_0^2 - 1206m_1^2m_0 \\
&\quad -108m_3^2m_0 \\
N1s018 &= +3840m_2m_0 - 792m_1m_3m_0 + 48m_4m_0 + 16284m_0 \\
&\quad +666m_1^2 + 54m_3^2 - 1224m_2 + 414m_1m_3 - 24m_4 - 4656 \\
N1s019 &= 408m_0^2 - 528m_2m_0 - 1044m_0 - 234m_1^2 + 9m_3^2 \\
&\quad +528m_2 - 18m_1m_3 + 636 \\
N1s020 &= 7344m_0^3 - 1920m_2m_0^2 - 23892m_0^2 - 1080m_1^2m_0 \\
&\quad -108m_3^2m_0 + 4032m_2m_0 - 756m_1m_3m_0 \\
N1s021 &= +24m_4m_0 + 25824m_0 + 1206m_1^2 + 108m_3^2 - 2112m_2 \\
&\quad +792m_1m_3 - 24m_4 - 9420 \\
N1s022 &= 1476m_0^4 - 288m_2m_0^3 - 7692m_0^3 + 2025m_1^2m_0^2 + 81m_3^2m_0^2 \\
N1s023 &= +1680m_2m_0^2 + 810m_1m_3m_0^2 + 12m_4m_0^2 + 14616m_0^2 \\
&\quad -4140m_1^2m_0 \\
N1s024 &= -144m_3^2m_0 - 2544m_2m_0 - 1548m_1m_3m_0 - 24m_4m_0 \\
&\quad -12972m_0 + 2124m_1^2 + 63m_3^2 + 1584m_2 + 738m_1m_3 \\
&\quad +12m_4 + 4752
\end{aligned}
\tag{3.45}
$$

$$
\begin{aligned}
N1s025 &= -5400m_0^2+96m_2m_0+10956m_0+540m_1^2+54m_3^2 \\
&\quad -240m_2+378m_1m_3-5556 \\
N1s026 &= -5832m_0^3+144m_2m_0^2+20916m_0^2-4050m_1^2m_0 \\
&\quad -162m_3^2m_0-672m_2m_0-1620m_1m_3m_0 \\
N1s027 &= +24m_4m_0-24480m_0+4140m_1^2+144m_3^2+528m_2 \\
&\quad +1548m_1m_3-24m_4+10224 \\
N1s028 &= -540m_0^4-1080m_2m_0^3+2724m_0^3+3288m_2m_0^2 \\
&\quad +192m_4m_0^2-5172m_0^2+1350m_1^2m_0+54m_3^2m_0 \\
&\quad -3216m_2m_0+540m_1m_3m_0 \\
N1s029 &= -408m_4m_0+3996m_0-1440m_1^2-54m_3^2+1584m_2 \\
&\quad -558m_1m_3+216m_4-972 \\
N1s030 &= 5184m_0^2+576m_2m_0-10884m_0+2025m_1^2+81m_3^2 \\
&\quad -1008m_2+810m_1m_3-36m_4+5592 \\
N1s031 &= 1296m_0^3+1512m_2m_0^2-6204m_0^2-3216m_2m_0 \\
&\quad -120m_4m_0+8832m_0-1350m_1^2-54m_3^2+1776m_2-540m_1m_3 \\
&\quad +120m_4-2808 \\
N1s032 &= -360m_0^3-720m_2m_0^2-180m_4m_0^2 \\
&\quad +1104m_0^2+1488m_2m_0+444m_4m_0
\end{aligned} \tag{3.46}
$$

$$
\begin{aligned}
N1s033 &= +96m_0 + 225m_1^2 + 9m_3^2 - 1200m_2 + 90m_1m_3 \\
&\quad -264m_4 - 1956 \\
N1s034 &= -648m_0^2 + 216m_2m_0 + 1932m_0 - 72m_2 - 72m_4 - 1068 \\
N1s035 &= 504m_0^2 + 288m_2m_0 + 72m_4m_0 - 1272m_0 - 288m_2 \\
&\quad -36m_4 - 1284 \\
N1s036 &= -60m_0^2 - 120m_2m_0 - 60m_4m_0 - 312m_0 - 528m_2 \\
&\quad +60m_4 + 1488 \\
N1s037 &= 72m_0 + 432m_2 + 108m_4 - 216 \\
N1s038 &= 24m_0 - 72m_2 - 36m_4 + 300 \\
N1s039 &= -216 \\
N1s040 &= 36
\end{aligned}
\tag{3.47}
$$

$$
\begin{aligned}
N1s2 = {} & N1s200 + N1s201r + N1s202r^2 + N1s203r^3 + N1s204r^4 \\
& +N1s205r^5 + N1s206r^6 + N1s207r^7 + N1s208r^8 + N1s209r^9 \\
& +N1s210r^{10} + N1s211r^{11} + N1s212r^{12} + N1s213r^{13} + N1s214r^{14} \\
& +N1s215r^{15} + N1s216r^{16} + N1s217r^{17} + N1s218r^{18} + N1s219r^{19} \\
& +N1s220r^{20} + N1s221r^{21} + N1s222r^{22} + N1s223r^{23} + N1s224r^{24} \\
& +N1s225r^{25} + N1s226r^{26} + N1s227r^{27}
\end{aligned}
\tag{3.48}
$$

$$
\begin{aligned}
N1s200 &= 0 \\
N1s201 &= 0 \\
N1s202 &= 72m_0^4 - 288m_0^3 + 432m_0^2 - 288m_0 + 72 \\
N1s203 &= 0 \\
N1s204 &= -168m_0^3 + 504m_0^2 - 504m_0 + 168 \\
N1s205 &= 180m_0^4 - 720m_0^3 + 1080m_0^2 - 720m_0 + 180 \\
N1s206 &= 120m_0^2 - 240m_0 + 120 \\
N1s207 &= -432m_0^3 + 1296m_0^2 - 1296m_0 + 432 \\
N1s208 &= -684m_0^4 + 2880m_0^3 - 4536m_0^2 + 3144m_0 - 804 \\
N1s209 &= 360m_0^2 - 720m_0 + 360 \\
N1s210 &= 1296m_0^3 - 4080m_0^2 + 4272m_0 - 1488 \\
N1s211 &= -1188m_0^4 + 4680m_0^3 - 6912m_0^2 + 4392m_0 - 972 \\
N1s212 &= -504m_0^2 + 1056m_0 - 552 \\
N1s213 &= 1536m_0^3 - 4536m_0^2 + 4464m_0 - 1428
\end{aligned}
\tag{3.49}
$$

$$\begin{aligned}
N1s214 &= 1620m_0^4 - 7632m_0^3 + 13248m_0^2 - 10224m_0 + 2988 \\
N1s215 &= -120m_0^2 + 312m_0 - 192 \\
N1s216 &= -4392m_0^3 + 14184m_0^2 - 15216m_0 + 5460 \\
N1s217 &= 1080m_0^3 - 3492m_0^2 + 4128m_0 - 1788 \\
N1s218 &= 3168m_0^2 - 6336m_0 + 3168 \\
N1s219 &= 2160m_0^3 - 8808m_0^2 + 11208m_0 - 5172 \\
N1s220 &= 180m_0^2 - 36 \\
N1s221 &= -3024m_0^2 + 6504m_0 - 3444 \\
N1s222 &= 1440m_0^2 - 3168m_0 + 972 \\
N1s223 &= 1224 - 432m_0 \\
N1s224 &= 396 - 576m_0 \\
N1s225 &= 240m_0 + 1056 \\
N1s226 &= -864 \\
N1s227 &= 144
\end{aligned} \tag{3.50}$$

$$\begin{aligned}
denom = {} & 6\left(15m_0^2 - 32m_0 + 17\right)r^6 + 6\left(-m_0^2 + 2m_0 - 1\right)r^3 \\
& +6\left(-2m_0^2 + 4m_0 - 2\right) + 6(5m_0 - 5)r^9 + 6(6 - 6m_0)r^8 \\
& +6(4m_0 - 4)r^5 + 6(2m_0 - 2)r^2 + 18r^{11} - 54r^{10} - 18r^7
\end{aligned} \tag{3.51}$$

Solving

$$\vec{v}(t_0) \cdot \vec{v}(t_0) = (\vec{v}(t_0) \cdot \hat{H}(t_0))^2 + (\vec{v}(t_0) \cdot \hat{R}(t_0))^2 + (\vec{v}(t_0) \cdot \hat{T}(t_0))^2 \tag{3.52}$$

we get

$$\vec{v}(t_0) \cdot \hat{H}(t_0) = -\sqrt{D} \tag{3.53}$$

or

$$\vec{v}(t_0) \cdot \hat{H}(t_0) = \sqrt{D} \tag{3.54}$$

where

$$
\begin{aligned}
D = &-\frac{(\vec{r}(t_0)\cdot\vec{v}(t_0))^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}+\frac{m_1\vec{r}(t_0)\cdot\vec{v}(t_0)(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}+\frac{m_3\vec{r}(t_0)\cdot\vec{v}(t_0)(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}\\
&-\frac{m_1^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}{16\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}-\frac{m_3^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}{16\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}-\frac{m_1m_3(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}\\
&-\frac{1}{16}(\vec{r}(t_0)\cdot\vec{r}(t_0))^2-\frac{(\vec{r}(t_0)\cdot\vec{v}(t_0))^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}+\frac{7m_1\vec{r}(t_0)\cdot\vec{v}(t_0)(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}\\
&+\frac{m_3\vec{r}(t_0)\cdot\vec{v}(t_0)(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}-\frac{3m_1^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}-\frac{3m_1m_3(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}\\
&+\frac{1}{8}m_0\vec{r}(t_0)\cdot\vec{r}(t_0)+\frac{1}{8}m_2\vec{r}(t_0)\cdot\vec{r}(t_0)-\frac{1}{4}\vec{v}(t_0)\cdot\vec{v}(t_0)\vec{r}(t_0)\cdot\vec{r}(t_0)-\frac{1}{8}\vec{r}(t_0)\cdot\vec{r}(t_0)\\
&-\frac{3m_0(\vec{r}(t_0)\cdot\vec{v}(t_0))^2\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}+\frac{3(\vec{r}(t_0)\cdot\vec{v}(t_0))^2\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}\\
&+\frac{3m_0m_1\vec{r}(t_0)\cdot\vec{v}(t_0)\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}-\frac{3m_1\vec{r}(t_0)\cdot\vec{v}(t_0)\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}\\
&+\frac{3m_0m_3\vec{r}(t_0)\cdot\vec{v}(t_0)\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}-\frac{3m_3\vec{r}(t_0)\cdot\vec{v}(t_0)\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}+\frac{1}{8}\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}-\frac{m_0^2}{16}\\
&-\frac{m_2^2}{16}-\frac{(\vec{r}(t_0)\cdot\vec{v}(t_0))^2}{16\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}-\frac{1}{4}(\vec{v}(t_0)\cdot\vec{v}(t_0))^2+\frac{m_0}{8}-\frac{m_0m_2}{8}+\frac{m_2}{8} \qquad (3.55)\\
&+\frac{3m_1\vec{r}(t_0)\cdot\vec{v}(t_0)}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}+\frac{1}{4}m_0\vec{v}(t_0)\cdot\vec{v}(t_0)+\frac{1}{4}m_2\vec{v}(t_0)\cdot\vec{v}(t_0)+\frac{3}{4}\vec{v}(t_0)\cdot\vec{v}(t_0)\\
&-\frac{9m_1^2}{16\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}-\frac{1}{16}-\frac{m_2}{8\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}+\frac{\vec{v}(t_0)\cdot\vec{v}(t_0)}{4\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}\\
&-\frac{3m_0(\vec{r}(t_0)\cdot\vec{v}(t_0))^2}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2\vec{r}(t_0)\cdot\vec{r}(t_0)}+\frac{3(\vec{r}(t_0)\cdot\vec{v}(t_0))^2}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2\vec{r}(t_0)\cdot\vec{r}(t_0)}\\
&+\frac{9m_0m_1\vec{r}(t_0)\cdot\vec{v}(t_0)}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2\vec{r}(t_0)\cdot\vec{r}(t_0)}-\frac{9m_1\vec{r}(t_0)\cdot\vec{v}(t_0)}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2\vec{r}(t_0)\cdot\vec{r}(t_0)}-\frac{1}{16\vec{r}(t_0)\cdot\vec{r}(t_0)}\\
&-\frac{m_0^2}{8(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}+\frac{m_0}{4(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}-\frac{m_0m_2}{8(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}+\frac{m_2}{8(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}+\frac{m_0\vec{v}(t_0)\cdot\vec{v}(t_0)}{4(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}\\
&-\frac{\vec{v}(t_0)\cdot\vec{v}(t_0)}{4(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}-\frac{1}{8(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}-\frac{9m_0^2(\vec{r}(t_0)\cdot\vec{v}(t_0))^2}{16\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^2}\\
&+\frac{9m_0(\vec{r}(t_0)\cdot\vec{v}(t_0))^2}{8\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^2}-\frac{9(\vec{r}(t_0)\cdot\vec{v}(t_0))^2}{16\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2(\vec{r}(t_0)\cdot\vec{r}(t_0))^2}-\frac{m_0}{8(\vec{r}(t_0)\cdot\vec{r}(t_0))^2}\\
&+\frac{1}{8(\vec{r}(t_0)\cdot\vec{r}(t_0))^2}-\frac{m_0^2}{16(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}+\frac{m_0}{8(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}-\frac{1}{16(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}
\end{aligned}
$$

$$\vec{r} = -\sqrt{E}\hat{H}(t_0)$$
$$-\frac{\hat{T}(t_0)\left(-3(m_0-1)\vec{r}(t_0)\cdot\vec{v}(t_0)+(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{5}{2}}(3m_1+m_3-6\vec{r}(t_0)\cdot\vec{v}(t_0))+\vec{r}(t_0)\cdot\vec{r}(t_0)(m_1+3\vec{r}(t_0)\cdot\vec{v}(t_0))\right)}{4\vec{r}(t_0)\cdot\vec{r}(t_0)\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)}$$
$$+\frac{1}{2}\hat{R}(t_0)(-m_0+\vec{r}(t_0)\cdot\vec{r}(t_0)+1) \tag{3.56}$$

where

$$E$$
$$= -\frac{\left(-3(m_0-1)\vec{r}(t_0)\cdot\vec{v}(t_0)+(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{5}{2}}(3m_1+m_3-6\vec{r}(t_0)\cdot\vec{v}(t_0))+\vec{r}(t_0)\cdot\vec{r}(t_0)(m_1+3\vec{r}(t_0)\cdot\vec{v}(t_0))\right)^2}{16(\vec{r}(t_0)\cdot\vec{r}(t_0))^2\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}$$
$$-\frac{1}{4}(-m_0+\vec{r}(t_0)\cdot\vec{r}(t_0)+1)^2+\vec{r}(t_0)\cdot\vec{r}(t_0) \tag{3.57}$$

$$\vec{v} = -\sqrt{F}\hat{H}(t_0)+G\hat{R}(t_0)+H\hat{T}(t_0) \tag{3.58}$$

or

$$\sqrt{F}\hat{H}(t_0)+G\hat{R}(t_0)+H\hat{T}(t_0) \tag{3.59}$$

where

$$F$$
$$= -\frac{\left(-3(m_0-1)\vec{r}(t_0)\cdot\vec{v}(t_0)+(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{5}{2}}(m_1+m_3-2\vec{r}(t_0)\cdot\vec{v}(t_0))+\vec{r}(t_0)\cdot\vec{r}(t_0)(3m_1-\vec{r}(t_0)\cdot\vec{v}(t_0))\right)^2}{16(\vec{r}(t_0)\cdot\vec{r}(t_0))^2\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)^2}$$
$$-\frac{\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}(m_0+m_2-2\vec{v}(t_0)\cdot\vec{v}(t_0)-1)+m_0-(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{5}{2}}+\vec{r}(t_0)\cdot\vec{r}(t_0)-1\right)^2}{16(\vec{r}(t_0)\cdot\vec{r}(t_0))^3}+\vec{v}(t_0)\cdot\vec{v}(t_0) \tag{3.60}$$

$$
\begin{aligned}
G = & \frac{3m_0\vec{r}(t_0)\cdot\vec{v}(t_0)}{4\left(\vec{r}(t_0)\cdot\vec{r}(t_0)-(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{5}{2}}\right)}+\frac{m_1(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)} \\
& +\frac{3m_1}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)}+\frac{m_3(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)} \\
& -\frac{(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}\vec{r}(t_0)\cdot\vec{v}(t_0)}{2\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)}-\frac{\vec{r}(t_0)\cdot\vec{v}(t_0)}{4\left((\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}-1\right)} \\
& -\frac{3\vec{r}(t_0)\cdot\vec{v}(t_0)}{4\left(\vec{r}(t_0)\cdot\vec{r}(t_0)-(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{5}{2}}\right)}
\end{aligned}
\tag{3.61}
$$

and

$$
\begin{aligned}
H = & -\frac{m_0}{4(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}-\frac{m_0}{4}-\frac{m_2}{4}+\frac{1}{4}\vec{r}(t_0)\cdot\vec{r}(t_0) \\
& -\frac{1}{4\sqrt{\vec{r}(t_0)\cdot\vec{r}(t_0)}}+\frac{1}{4(\vec{r}(t_0)\cdot\vec{r}(t_0))^{\frac{3}{2}}}+\frac{1}{2}\vec{v}(t_0)\cdot\vec{v}(t_0)+\frac{1}{4}
\end{aligned}
\tag{3.62}
$$

4 Contact

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