# Definition and Existence of the Eigenderivative

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**Abstract**

We define the eigenderivatives of a linear operator on any real or complex Banach space.

**Keywords.** absolute convergence, Banach space, eigenderivative, eigenvalue, eigenvector, Hilbert space, linear operator.

## 1 Motivation

If $K \in \mathrm{End}(X)$[1] is a linear operator on $X$, a real or complex Banach space, we hypothesize that the eigenvectors and eigenvalues of an operator "nearly equal" to $K$ will typically be "nearly equal" to those of $K$. In this paper we explore this question and give a definition of the "eigenderivatives," i.e. the "rates of change" of eigenvectors and eigenvalues.

[1] The notation $\mathrm{End}(X)$ denotes the linear operators mapping $X$ into itself.

## 2 Definition of Eigenderivatives

**Definition** Let $X$ be a real or complex Banach space with dimension $N \leqslant \infty$, and let $K \in \mathrm{End}\,(X)$ be a linear operator. Let $\{e_i\}_{i=1}^{M}$ be a linearly independent collection of unit eigenvectors of $K$ with distinct eigenvalues $\{\lambda_i\}_{i=1}^{M}$. The collection $\{e_i\}_{i=1}^{M}$ may comprise some or all of the unit eigenvectors of $K$. Let $S = \langle e_i \rangle_{i=1}^{M}$ denote the subspace spanned by $\{e_i\}_{i=1}^{M}$. Let $F = \mathbb{C}$ or $\mathbb{R}$ identify the field over $X$. Suppose $J \in \mathrm{End}\,(S)$. Finally, suppose for each $i \in \{1, 2, 3, \cdots, M\}$[2],

$$\sum_{\substack{j=1 \\ j \neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i - \lambda_j} e_j \quad (2.1)$$

and

$$\sum_{\substack{j=1 \\ j \neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i - \lambda_j} \left(J - \vartheta_{i,i}\right) e_j \quad (2.2)$$

converge absolutely, where $Je_i = \sum\limits_{j=1}^{M} \vartheta_{i,j} e_j$. The uniqueness of each $\vartheta_{i,j}$ is assured by the linear independence of $\{e_i\}_{i=1}^{M}$. Then for each $i \in \{1, 2, 3, \cdots, M\}$ we define $\Delta_i : \mathrm{End}\,(X) \times \mathrm{End}\,(S) \to X$ and $\Lambda_i : \mathrm{End}\,(S) \to F$ according to

[2]In case $M = \infty$, $\{1, 2, 3, \cdots, M\}$ means the set of positive integers.

$$\Lambda_i(J) = \vartheta_{i,i} \quad (2.3)$$

and

$$\Delta_i(K,J) = \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j} e_j. \quad (2.4)$$

**Proposition 2.1** *If* $\sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}(J-\vartheta_{i,i})e_j$ *converges absolutely for every* $i \in \{1,2,\ldots,M\}$, *then for any* $h \in F$,

$$\begin{aligned}(K+hJ)(e_i+h\Delta_i(K,J)) &\equiv (\lambda_i+h\Lambda_i(J))(e_i+h\Delta_i(K,J)) \\ &+h^2\sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}(J-\vartheta_{i,i})e_j\end{aligned} \quad (2.5)$$

*In particular, for small* $|h|$,

$$\begin{aligned}(K+hJ)(e_i+h\Delta_i(K,J)) &= (\lambda_i+h\Lambda_i(K,J))(e_i+h\Delta_i(K,J)) \\ &+\mathcal{O}\left(h^2\right)\end{aligned} \quad (2.6)$$

**Proof**

$$(K+hJ)(e_i+h\Delta_i(K,J)) - (\lambda_i+h\Lambda_i(J))(e_i+h\Delta_i(K,J))$$

$$
\begin{aligned}
&= Ke_i + hK\Delta_i(K,J) + hJe_i + h^2 J\Delta_i(K,J) - \lambda_i e_i - h\lambda_i\Delta_i(K,J) \\
&\quad - h\Lambda_i(J)\,e_i - h^2\Lambda_i(J)\,\Delta_i(K,J) - h^2 \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}\left(J-\vartheta_{i,i}\right)e_j \\
&= hK\Delta_i(K,J) + hJe_i + h^2 J\Delta_i(K,J) - h\lambda_i\Delta_i(K,J) - h\Lambda_i(J)\,e_i \\
&\quad - h^2\Lambda_i(J)\,\Delta_i(K,J) - h^2 \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}\left(J-\vartheta_{i,i}\right)e_j \\
&= hK\Delta_i(K,J) + hJe_i + h^2 J\Delta_i(K,J) - h\lambda_i\Delta_i(K,J) - h\vartheta_{i,i}e_i \\
&\quad - h^2\vartheta_{i,i}\Delta_i(K,J) - h^2 \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}\left(J-\vartheta_{i,i}\right)e_j \\
&= hK\Delta_i(K,J) + hJe_i + h^2 J\Delta_i(K,J) - h\lambda_i\Delta_i(K,J) - h\vartheta_{i,i}e_i \\
&\quad - h^2\vartheta_{i,i}\Delta_i(K,J) - h^2 \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}Je_j + h^2\vartheta_{i,i} \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}e_j \\
&= hK\Delta_i(K,J) + hJe_i + h^2 J\Delta_i(K,J) - h\lambda_i\Delta_i(K,J) - h\vartheta_{i,i}e_i \\
&\quad - h^2 \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}Je_j \\
&= hK\Delta_i(K,J) + hJe_i - h\lambda_i\Delta_i(K,J) - h\vartheta_{i,i}e_i \\
&= hK \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}e_j + hJe_i - h\lambda_i \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}e_j - h\vartheta_{i,i}e_i \\
&= h \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}\lambda_j e_j + hJe_i - h\lambda_i \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}e_j - h\vartheta_{i,i}e_i
\end{aligned}
$$

$$= -h\sum_{\substack{j=1\\j\neq i}}^{M}\vartheta_{i,j}e_j + hJe_i - h\vartheta_{i,i}e_i$$

$$= -h\sum_{\substack{j=1\\j\neq i}}^{M}\vartheta_{i,j}e_j + h\sum_{j=1}^{M}\vartheta_{i,j}e_j - h\vartheta_{i,i}e_i$$

$$= 0. \quad (2.7)$$

**Definition** We call $\Delta_i : \mathrm{End}\,(X) \times \mathrm{End}\,(S) \to S$ and $\Lambda_i : \mathrm{End}\,(S) \to F$ the eigenderivatives of $K$ with respect to $J$

**Proposition 2.2** *If $J$ is bounded, and if for each $i \in \{1, 2, 3, \cdots, M\}$ we have that $\sup_j |\vartheta_{i,j}| < \infty$ and $\sum_{\substack{j=1\\j\neq i}}^{M} \frac{1}{\lambda_i - \lambda_j}$ converges absolutely for every $i = 1, 2, \ldots M$, then the eigenderivatives exist.*

**Proof**

$$\begin{aligned}\sum_{\substack{j=1\\j\neq i}}^{M}\left\|\frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}e_j\right\| &= \sum_{\substack{j=1\\j\neq i}}^{M}\left|\frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}\right|\|e_j\| = \sum_{\substack{j=1\\j\neq i}}^{M}\left|\frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}\right| \\ &= \sum_{\substack{j=1\\j\neq i}}^{M}|\vartheta_{i,j}|\frac{1}{|\lambda_i-\lambda_j|} \leqslant \left(\sup_j|\vartheta_{i,j}|\right)\sum_{\substack{j=1\\j\neq i}}^{M}\frac{1}{|\lambda_i-\lambda_j|} < \infty\end{aligned} \quad (2.8)$$

hence $\sum_{\substack{j=1\\j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}e_j$ converges absolutely for each $i = 1, 2, \ldots, M$.

$$\sum_{\substack{j=1\\j\neq i}}^{M}\left\|\frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}\left(J-\vartheta_{i,i}\right)e_j\right\| = \sum_{\substack{j=1\\j\neq i}}^{M}\left\|\frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}\left(Je_j-\vartheta_{i,i}e_j\right)\right\|$$

$$
\begin{aligned}
&\leqslant \sum_{\substack{j=1\\ j\neq i}}^{M} \left\| \frac{\vartheta_{i,j}}{\lambda_i - \lambda_j} J e_j \right\| + \sum_{\substack{j=1\\ j\neq i}}^{M} \left\| \frac{\vartheta_{i,j}\vartheta_{i,i}}{\lambda_i - \lambda_j} e_j \right\| \\
&= \sum_{\substack{j=1\\ j\neq i}}^{M} \left| \frac{\vartheta_{i,j}}{\lambda_i - \lambda_j} \right| \|J e_j\| + \sum_{\substack{j=1\\ j\neq i}}^{M} \left| \frac{\vartheta_{i,j}\vartheta_{i,i}}{\lambda_i - \lambda_j} \right| \|e_j\| \\
&\leqslant \sum_{\substack{j=1\\ j\neq i}}^{M} |\vartheta_{i,j}| \frac{1}{|\lambda_i - \lambda_j|} \|J\| \, \|e_j\| + |\vartheta_{i,i}| \sum_{\substack{j=1\\ j\neq i}}^{M} |\vartheta_{i,j}| \frac{1}{|\lambda_i - \lambda_j|} \\
&= \|J\| \sum_{\substack{j=1\\ j\neq i}}^{M} |\vartheta_{i,j}| \frac{1}{|\lambda_i - \lambda_j|} + |\vartheta_{i,i}| \sum_{\substack{j=1\\ j\neq i}}^{M} |\vartheta_{i,j}| \frac{1}{|\lambda_i - \lambda_j|} \\
&\leqslant \|J\| \sup_j |\vartheta_{i,j}| \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{1}{|\lambda_i - \lambda_j|} + \left( \sup_j |\vartheta_{i,j}| \right)^2 \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{1}{|\lambda_i - \lambda_j|} \\
&= \left( \|J\| + \sup_j |\vartheta_{i,j}| \right) \left( \sup_j |\vartheta_{i,j}| \right) \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{1}{|\lambda_i - \lambda_j|} < \infty
\end{aligned}
\tag{2.9}
$$

hence $\sum\limits_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta_{i,j}}{\lambda_i - \lambda_j} \left( J - \vartheta_{i,i} \right) e_j$ converges absolutely.

<u>Remark</u> In the special case $M = 2$, $J$ is necessarily bounded and we always have $\sup\limits_j |\vartheta_{i,j}| < \infty$, thus the eigenderivatives exist, and

$$
\Delta_1(K, J) = \frac{\vartheta_{1,2}}{\lambda_1 - \lambda_2} e_2
$$

$$
\Lambda_1(J) = \vartheta_{1,1},
$$

$$\Delta_2(K,J) = -\frac{\vartheta_{2,1}}{\lambda_1-\lambda_2}e_1,$$

and

$$\Lambda_2(J) = \vartheta_{2,2}.$$

# 3 Examples

**Example** Take $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -7 \end{pmatrix}$ and $J = \begin{pmatrix} -1 & -2 & 3 \\ 3 & -4 & 7 \\ 4 & 5 & -2 \end{pmatrix}$.

The eigenvectors of $K$ are given by

$e = ((0,0,1),(0,1,0),(1,0,0)$.

The eigenvvalues of $K$ are given by

$\lambda = \{-7,2,1\}$

The choice of $J$ implies $\vartheta = \{\{-2,7,3\},\{5,-4,-2\},\{4,3,-1\}\}$. The eigenderivatives are then

$\Delta = \left\{\left\{-\frac{3}{8},-\frac{7}{9},0\right\},\left\{-2,0,\frac{5}{9}\right\},\left\{0,-3,\frac{1}{2}\right\}\right\}$ and

$\Lambda = \{-2,-4,-1\}$.

$$\begin{pmatrix} \frac{1}{72}h(189+139h) \\ \frac{1}{72}h(392+143h) \\ -7-2h-\frac{97h^2}{18} \end{pmatrix} = \begin{pmatrix} -\frac{3}{8}(-7-2h)h \\ -\frac{7}{9}(-7-2h)h \\ -7-2h \end{pmatrix} + \begin{pmatrix} \frac{85h^2}{72} \\ \frac{31h^2}{72} \\ -\frac{97h^2}{18} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{3}h(-12+11h) \\ 2-4h-\frac{19h^2}{9} \\ \frac{2}{9}(5-41h)h \end{pmatrix} = \begin{pmatrix} -2(2-4h)h \\ 2-4h \\ \frac{5}{9}(2-4h)h \end{pmatrix} + \begin{pmatrix} -\frac{13h^2}{3} \\ -\frac{19h^2}{9} \\ -\frac{62h^2}{9} \end{pmatrix}$$

$$\begin{pmatrix} 1-h+\frac{15h^2}{2} \\ \frac{1}{2}h(-6+31h) \\ \frac{1}{2}(1-32h)h \end{pmatrix} = \begin{pmatrix} 1-h \\ -3(1-h)h \\ \frac{1}{2}(1-h)h \end{pmatrix} + \begin{pmatrix} \frac{15h^2}{2} \\ \frac{25h^2}{2} \\ -\frac{31h^2}{2} \end{pmatrix}$$

In the following examples, let $X$ be the completion of the linear space spanned by the simple functions $f_n : (0,\infty) \to \mathbb{R}$ verifying $f_n(x) = \begin{cases} 1 & n \leqslant x < n+1 \\ 0 & \text{elsewhere} \end{cases}$ for positive integers $n$, with $F = \mathbb{R}$. Let the $\|\cdot\|$ be given by the $L^2$ norm, i.e. $\|f\| = \sqrt{\int_0^\infty f(x)^2\,dx}$

**Example** An unbounded operator, for which eigenderivatives exist.

Let $K$ satisfy $Kf_n(x) = nf_n(x)$. Then $\lambda_n = n$ and $e_n = f_n$. Take $S = \langle f_n \rangle_{n=1}^{\infty}$. Let $J$ be the operator satisfying $Jf_n(x) = \sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{n+k}} f_k(x)$.

Note that $\|Jf_n(x)\| = \infty$ thus $J$ is unbounded. Next note that

$$\sum_{\substack{j=1\\j\neq i}}^{\infty} \left\| \frac{\vartheta_{i,j}}{\lambda_i - \lambda_j} e_j \right\| = \sum_{\substack{j=1\\j\neq i}}^{\infty} \left\| \frac{1}{i-j} \frac{1}{\sqrt{i+j}} f_j(x) \right\| = \sum_{\substack{j=1\\j\neq i}}^{\infty} \frac{1}{|i-j|} \frac{1}{\sqrt{i+j}} < \infty$$

Next,

$$
\begin{aligned}
&\left\| \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{\vartheta_{i,j}}{\lambda_i-\lambda_j}\left(J-\vartheta_{i,i}\right) e_j \right\| \\
&= \left\| \sum_{\substack{j=1\\ j\neq i}}^{\infty} \left( \frac{1}{\sqrt{2i}}\frac{1}{j-i}\frac{1}{\sqrt{i+j}} + \sum_{\substack{k=1\\ k\neq i}}^{\infty} \frac{1}{\sqrt{k+j}}\frac{1}{i-j}\frac{1}{\sqrt{i+j}} f_k(x) \right) f_j\left(x\right) \right\| \\
&= \left\| \sum_{\substack{j=1\\ j\neq i}}^{\infty} \left( \frac{1}{\sqrt{2i}}\frac{1}{j-i}\frac{1}{\sqrt{i+j}} f_j\left(x\right) + \sum_{\substack{k=1\\ k\neq i}}^{\infty} \frac{1}{\sqrt{k+j}}\frac{1}{i-j}\frac{1}{\sqrt{i+j}} f_k(x) f_j\left(x\right) \right) \right\| \\
&= \left\| \sum_{\substack{j=1\\ j\neq i}}^{\infty} \left( \frac{1}{\sqrt{2i}}\frac{1}{j-i}\frac{1}{\sqrt{i+j}} f_j\left(x\right) + \frac{1}{\sqrt{j+j}}\frac{1}{i-j}\frac{1}{\sqrt{i+j}} f_j\left(x\right)^2 \right) \right\| \\
&= \left\| \sum_{\substack{j=1\\ j\neq i}}^{\infty} \left( \frac{1}{\sqrt{2i}}\frac{1}{j-i}\frac{1}{\sqrt{i+j}} + \frac{1}{\sqrt{2j}}\frac{1}{i-j}\frac{1}{\sqrt{i+j}} \right) f_j\left(x\right) \right\| \\
&= \left\| \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{1}{j-i}\frac{1}{\sqrt{i+j}} \left( \frac{1}{\sqrt{2i}} - \frac{1}{\sqrt{2j}} \right) f_j\left(x\right) \right\| \leqslant \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{1}{\sqrt{i+j}}\frac{1}{j-i}\frac{\sqrt{2j}-\sqrt{2i}}{2\sqrt{ij}} \left\| f_j\left(x\right) \right\| \\
&= \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{1}{\sqrt{i+j}}\frac{1}{\sqrt{j}+\sqrt{i}}\frac{1}{\sqrt{2ij}} = \frac{1}{\sqrt{2i}} \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{1}{\sqrt{i+j}}\frac{1}{\sqrt{j}+\sqrt{i}}\frac{1}{\sqrt{j}} < \infty
\end{aligned}
\tag{3.1}
$$

Thus the requirements of the Definition are satisfied, and the eigenderivatives exist. They are $\Lambda_i\left(J\right) = \frac{1}{\sqrt{2i}}$ and $\Delta_i\left(K,J\right) = \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{1}{(i-j)\sqrt{i+j}} f_j\left(x\right)$.

**Example** A bounded operator, for which eigenderivatives exist. Take $S = \langle f_n \rangle_{n=1}^{\infty}$. Let $K$ satisfy $K f_n(x) = \frac{1}{n} f_n(x)$. Then $\lambda_n = \frac{1}{n}$ and $e_n = f_n$. Let $J$ be the operator satisfying $J f_n(x) = \sum\limits_{k=1}^{\infty} \frac{1}{(n+k)^2} f_k(x)$. Note that $\|J f_n(x)\| \leqslant \sum\limits_{k=1}^{\infty} \frac{1}{(n+k)^2} < \frac{\pi^2}{6} < \infty$ thus $J$ is bounded. Next note that

$$\left\| \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{\vartheta_{i,j}}{\lambda_i - \lambda_j} e_j \right\| \leqslant \sum_{\substack{j=1\\ j\neq i}}^{\infty} \left| \frac{\frac{1}{(i+j)^2}}{\frac{1}{i} - \frac{1}{j}} \right| \leqslant i \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{1}{(i+j)^2} \left| \frac{j}{j-i} \right| \leqslant$$

$$i \sum_{j=1}^{i-1} \frac{1}{(i+j)^2} \frac{j}{i-j} + i \sum_{j=i+1}^{\infty} \frac{1}{(i+j)^2} \frac{j}{j-i} \leqslant i \sum_{j=1}^{i-1} \frac{1}{(i+j)^2} \frac{j}{i-j} + i(i+1) \sum_{j=i+1}^{\infty} \frac{1}{(i+j)^2} <$$

$$i \sum_{j=1}^{i-1} \frac{1}{(i+j)^2} \frac{j}{i-j} + \frac{\pi^2}{6} i(i+1) < \infty$$

and

$$\left\| \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{\vartheta_{i,j}}{\lambda_i - \lambda_j} \left(J - \vartheta_{i,i}\right) e_j \right\| = \left\| \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{\vartheta_{i,j}}{\lambda_i - \lambda_j} \left(J e_j - \vartheta_{i,i} e_j\right) \right\| =$$

$$\left\| \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{\frac{1}{(i+j)^2}}{\frac{1}{i} - \frac{1}{j}} \left( -\frac{1}{(2i)^2} f_j(x) + \sum_{k=1}^{\infty} \frac{1}{(j+k)^2} f_k(x) \right) \right\| =$$

$$\left\| \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{1}{(i+j)^2} \frac{ij}{j-i} \left( -\frac{1}{(2i)^2} f_j(x) + \frac{1}{(2j)^2} f_j(x) + \sum_{\substack{k=1\\ k\neq j}}^{\infty} \frac{1}{(j+k)^2} f_k(x) \right) \right\| =$$

$$\left\| \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{1}{(i+j)^2} \frac{ij}{j-i} \left( \left( \frac{1}{(2j)^2} - \frac{1}{(2i)^2} \right) f_j(x) + \sum_{\substack{k=1\\ k\neq j}}^{\infty} \frac{1}{(j+k)^2} f_k(x) \right) \right\| \leqslant$$

$$\left\| \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{1}{(i+j)^2} \frac{ij}{j-i} \left( \frac{1}{(2j)^2} - \frac{1}{(2i)^2} \right) f_j(x) \right\| + \left\| \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{1}{(i+j)^2} \frac{ij}{j-i} \sum_{\substack{k=1\\ k\neq j}}^{\infty} \frac{1}{(j+k)^2} f_k(x) \right\| \leqslant$$

$$\sum_{\substack{j=1\\ j\neq i}}^{\infty} \left| \frac{1}{(i+j)^2} \frac{ij}{j-i} \left( \frac{1}{(2j)^2} - \frac{1}{(2i)^2} \right) \right| + i \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{1}{(i+j)^2} \left| \frac{j}{j-i} \right| \sum_{\substack{k=1\\ k\neq j}}^{\infty} \frac{1}{(j+k)^2} \leqslant \frac{1}{8} + \frac{\pi^2\left(\pi^2-3\right)}{72} i < \infty$$

Thus the requirements of the Definition are satisfied, and the eigenderivatives exist. Then $\lambda_n = \frac{1}{n}$ and $e_n = f_n$. Let $J$ be the operator satisfying $Jf_n(x) = \sum\limits_{k=1}^{\infty} \frac{1}{(n+k)^2} f_k(x)$. They are $\Lambda_i(J) = \frac{1}{(2i)^2}$ and $\Delta_i(K,J) = i \sum\limits_{\substack{j=1 \\ j \neq i}}^{\infty} \frac{j}{j-i} \frac{1}{(i+j)^2} f_j(x)$. In this case we have $\|\Delta_i(K,J)\| \leqslant i \sum\limits_{\substack{j=1 \\ j \neq i}}^{\infty} \frac{j}{|j-i|} \frac{1}{(i+j)^2} \leqslant \frac{1}{12}(\pi^2 - 3)\, i$

# 4 Eigenderivatives Are Operators

$$\Lambda_i(\alpha_1 J_1 + \alpha_2 J_2) = \alpha_1 \Lambda_i(J_1) + \alpha_2 \Lambda_i(J_2) \quad (4.1)$$

$$\Delta_i(K, \alpha_1 J_1 + \alpha_2 J_2) = \alpha_1 \Delta_i(K, J_1) + \alpha_2 \Delta_i(K, J_2) \quad (4.2)$$

thus the maps $L_1 : J \mapsto \Lambda_i(J)$ and $L_2 : J \mapsto \Delta_i(K,J)$ are linear operators on $S$. $L_1$ is a linear functional on $S$, and $L_2$ is a member of $\mathrm{End}(S)$.

# 5 BIBLIOGRAPHY

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